CS 173, Spring 18 Examlet 8, Part A

Name:____

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(20 points) Let x be a non-zero real number such that $x + \frac{1}{x}$ is an integer. Use (strong) induction to prove that $x^n + \frac{1}{x^n}$ is an integer, for any natural number n.

Hint: $(a^n + b^n)(a + b) = (a^{n+1} + b^{n+1}) + ab(a^{n-1} + b^{n-1})$, for any real numbers a and b.

Solution: Let x be a non-zero real number such that $x + \frac{1}{x}$.

Proof by induction on n.

Base case(s): At $n = 0$, $x^n + \frac{1}{x^n} = 1 + 1 = 2$, which is an integer for any non-zero x.

At $n = 1$, $x^n + \frac{1}{x^n} = x + \frac{1}{x}$, so the claim is obviously true.

[Notice that we need two base cases because our inductive step will use the result at two previous values of n.]

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $x^n + \frac{1}{x^n}$ is an integer, for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

Using the hint, we get

$$x^{k+1} + \frac{1}{x^{k+1}} = (x^k + \frac{1}{x^k})(1 + \frac{1}{x}) - (x \cdot \frac{1}{x})(x^{k-1} + \frac{1}{x^{k-1}}) = (x^k + \frac{1}{x^k})(1 + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}})$$

By the inductive hypothesis, $x^k + \frac{1}{x^k}$ and $x^{k-1} + \frac{1}{x^{k-1}}$ are integers. We were also given that $(1 + \frac{1}{x})$ is an integer. The righthand side must be an integer since it's made by multiplying and subtracting integers. So the lefthand side $x^{k+1} + \frac{1}{x^{k+1}}$ must also be an integer. This is what we needed to show.

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(20 points) Suppose that $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$h(1) = 1 \qquad h(2) = 7$

$h(n+1) = 7h(n) - 12h(n-1)$ for all $n \geq 2$

Use (strong) induction to prove that $h(n) = 4^n - 3^n$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $h(1) = 1$ and $4^n - 3^n = 4 - 3 = 1$. So the claim holds.

At $n = 2$, $h(2) = 7$ and $4^n - 3^n = 16 - 9 = 7$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $h(n) = 4^n - 3^n$ for $n = 1, 2, \ldots, k$.

Rest of the inductive step:

Using the definition of h and the inductive step, we get

$$\begin{aligned} h(k+1) &= 7h(k) - 12h(k-1) \\ &= 7(4^k - 3^k) - 12(4k - 1 + 3^{k-1}) \\ &= 7(4^k - 3^k) - (3 \cdot 4k + 4 \cdot 3^k) \\ &= (7-3)4^k - (7-4)3^k = 4^{k+1} - 3^{k+1} \end{aligned}$$

So $h(k+1) = 4^{k+1} - 3^{k+1}$, which is what we needed to show.

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(20 points) Recall that F_n is the nth Fibonacci number, and the positive Fibonacci numbers start with $F_1 = F_2 = 1$. Use (strong) induction to prove the following claim:

> Claim: Every positive integer can be written as the sum of (one or more) distinct Fibonacci numbers.

Hints: You can assume that the Fibonacci numbers are strictly increasing starting with F_1. To write x as the sum of Fibonacci numbers, start by including the largest Fibonacci number F_p such that $F_p \leq x$. (And therefore $x < F_{p+1}$.) How large is the remaining part of x?

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $n = 1 = F_1$. So n is the sum of a single Fibonacci number.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that n is the sum of (one or more) distinct Fibonacci numbers, for $n = 1, \ldots, k-1$.

Inductive Step: Consider k. Notice that we can assume that $k > 1$, since $n = 1$ was already covered in the base case. Let F_p be the largest Fibonacci number $\leq k$. There are two cases:

Case 1: $F_p = k$. Then k is the sum of a single Fibonacci number.

Case 2: $F_p < k$. Let $y = k - F_p$. Since F_p must be positive, y is less than k. So we can apply the inductive hypothesis to y. That is $y = F_{i_1} + \ldots + F_{i_j}$, where $F_{i_1} \ldots F_{i_j}$ are all distinct.

Notice that $x < F_{p+1} = P_p + P_{p-1}$ So $y = k - F_p < P_{p-1}$. This means that $F_{i_1} \ldots F_{i_j}$ are all smaller than F_p, so F_p can't be equal to any of them.

So then $k = y + F_p = (F_{i_1} + \ldots F_{i_j}) + F_p$ and the numbers in this sum are all distinct. So k is the sum of (one or more) distinct Fibonacci numbers, which is what we needed to prove.

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(20 points) Use (strong) induction to prove the following claim:

For any positive integer $n \geq 2$, if G is a graph with n nodes and more than $(n-1)(n-2)/2$ edges, then G is connected.

Hint: pick a node x. Perhaps x is connected to all the other nodes. If not, remove x to create a smaller graph H. What is the smallest number of edges that could remain in H? Notice that H has too few nodes to contain all the edges in G, so there is an edge from x to H.

Solution: Proof by induction on n.

Base case(s): $n = 2$ The graph has two nodes and one edge. There's only one such graph and it's connected.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: If G is a graph with n nodes and more than $(n-1)(n-2)/2$ edges, then G is connected, for $n = 2, \ldots, k-1$.

Rest of the inductive step: Let G be a graph with k nodes and more than $(k-1)(k-2)/2$ edges.

Pick a node x in G. Remove x (and its edges) to produce a smaller graph H.

Case 1: x is connected to all the other $k-1$ nodes in G. Then there is a path from any node to any other node, via x. So G is connected.

Case 2: x is connected to $k-2$ or fewer nodes. This means that H must have more than $(k-1)(k-2)/2 - (k-2)$ edges. $(k-1)(k-2)/2 - (k-2) = (k-1)(k-2)/2 - 2(k-2)/2 = (k-3)(k-2)/2$. So H has $k-1$ nodes and more than $(k-3)(k-2)/2$ edges. By the inductive hypothesis, H must be connected.

H has $k-1$ nodes. The maximum number of edges in H is $(k-1)(k-2)/2$, i.e. the number of edges in a complete graph. Since G has more edges than that, there must be at least one edge connecting x to a node of H.

Since H is connected, and x is connected to a node in H, the full graph G is connected, which is what we needed to prove.

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(20 points) Recall that F_n is the nth Fibonacci number, and these start with $F_0 = 0$, $F_1 = 1$. Use (strong) induction to prove the following claim:

Claim: $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for any positive integer n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $F_{n-1}F_{n+1} - (F_n)^2 = F_0F_2 - (F_1)^2 = 0 - 1 = -1 = (-1)^n$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $F_{n-1}F_{n+1} - (F_n)^2 = (-1)^n$ for $n = 1, \ldots, k$.

Inductive Step: By the definition of the Fibonacci numbers, $F_{k+2} = F_{k+1} + F_k$. So

$F_kF_{k+2} - (F_{k+1})^2 = F_k(F_{k+1} + F_k) - (F_{k+1})^2 = (F_k)^2 + F_kF_{k+1} - (F_{k+1})^2 = (F_k)^2 + F_{k+1}(F_k - F_{k+1})$

But since $F_{k+1} = F_k + F_{k-1}$, $F_k - F_{k+1} = -F_{k-1}$. So

$F_kF_{k+2} - (F_{k+1})^2 = (F_k)^2 - F_{k+1}F_{k-1} = (-1)(F_{k+1}F_{k-1} - (F_k)^2)$

By the inductive hypothesis, $F_{k-1}F_{k+1} - (F_k)^2 = (-1)^k$. Substituting this into the previous equation, we get

$F_kF_{k+2} - (F_{k+1})^2 = (-1)(-1)^k$

So $F_kF_{k+2} - (F_{k+1})^2 = (-1)^{k+1}$, which is what we needed to prove.

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(20 points) Let function $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$f(0) = 3$

$f(1) = 9$

$f(n) = f(n-1) + 2f(n-2)$, for $n \geq 2$

Use (strong) induction to prove that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for any natural number n.

Solution: Proof by induction on n.

Base case(s): For $n = 0$, we have $4 \cdot 2^0 + (-1)^{-1} = 4 - 1 = 3$ which is equal to $f(0)$. So the claim holds.

For $n = 1$, we have $4 \cdot 2^1 + (-1)^0 = 8 + 1 = 9$ which is equal to $f(1)$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $f(n) = 4 \cdot 2^n + (-1)^{n-1}$ for $n = 0, 1, \ldots, k-1$ where $k \geq 2$.

Rest of the inductive step:

$$
\begin{aligned}
f(k) &= f(k-1) + 2f(k-2) && \text{by definition of } f \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 2(4 \cdot 2^{k-2} + (-1)^{k-3}) && \text{by inductive hypothesis} \\
&= (4 \cdot 2^{k-1} + (-1)^{k-2}) \quad + \quad 4 \cdot 2^{k-1} + 2(-1)^{k-3} \\
&= 8 \cdot 2^{k-1} + (-1)^{k-2} - 2(-1)^{k-2} \\
&= 4 \cdot 2^k - (-1)^{k-2} \\
&= 4 \cdot 2^k + (-1)^{k-1}
\end{aligned}
$$

So $f(k) = 4 \cdot 2^k + (-1)^{k-1}$, which is what we needed to show.