CS 173, Spring 18 Examlet 9, Part A

Name:_____

NetID:_____ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(18 points) Recall that ϵ is shorthand for the empty (zero-length) string. Here is a grammar G, with start symbol S and terminal symbols a and b.

$$S \rightarrow a\ S\ b\ S \mid b\ S\ a\ S \mid \epsilon$$

Use (strong) induction to prove that any string with equal numbers of a's and b's can be generated by grammar G. That is, show how to build parse trees for these strings. You can use (without proof) this fact (and the similar fact with a and b swapped).

> Fact: Suppose the number of a's in a string w is one more than the number of b's in w. Then we can divide w into $w = xay$, where string x (and therefore also string y) has equal numbers of a's and b's.

The induction variable is named _____ and it is the _____ of/in the string.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step:

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(18 points) A Vintage tree is a binary tree in which each node X contains an integer label $v(X)$ such that

- If X is a leaf, $v(X)$ is 7, 23, or 31.
- If X has one child Y, then $v(X) = v(Y) + 7$.
- If X has two children Y and Z, then $v(X) = v(Y)v(Z)$.

Use strong induction to prove that the value in the root of a Vintage tree is always positive.

The induction variable is named _____ and it is the _____ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step:

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(18 points) Orange trees are binary trees whose nodes are labelled with strings, such that

- Each leaf node has label `tip`, `top`, or `tack`
- If a node has one child, it has label $\alpha\alpha$ where where α is the child's label. E.g. if the child has label `top` then the parent has `toptop`.
- If a node has two children, it contains $\alpha\beta$ where α and β are the child labels. E.g. if the children have labels `tip` and `top`, then the parent has label `tiptop`.

Let S(n) be the length of the label on node n. Let L(n) be the number of leaves in the subtree rooted at n. Use (strong) induction to prove that $S(n) \geq 3L(n)$ if n is the root node of any Orange tree.

The induction variable is named _____ and it is the _____ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step:

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(18 points) Recall that a node in a full binary tree is either a leaf or has exactly two children. A Illini tree is a full binary tree whose nodes contain a color (blue or orange) plus a positive integer, such that

- All leaf nodes have label $(\text{blue}, 1)$.
- If an internal node has label (orange, p), then both its children must have label (blue, p).
- If an internal node has label (blue, p), then its children must have label $(\text{blue}, p-1)$ or $(\text{orange}, p-1)$.

Use (strong) induction to prove that a Illini tree with root label (blue, p) or (orange, p) has at least $2^p - 1$ nodes.

The induction variable is named _____ and it is the _____ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step:

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(18 points) Lemon trees have nodes labelled with integer values such that:

- Every leaf node has label 5 or 7.
- An internal node with label 0 has exactly three children.
- An internal node with label 1 has exactly two children.

The "total value" of a Lemon tree is the sum of the labels on all its nodes. Use (strong) induction that the total value of any Lemon tree is odd. You may assume basic divisibility facts e.g. the sum of two odd numbers is even.

The induction variable is named ____ and it is the _____ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step:

Name:_____

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(18 points) Here is a grammar G, with start symbol S and terminal symbols a and b.

$$S \rightarrow a\ S\ b \mid b\ S\ a \mid S\ S \mid a\ b$$

Use (strong) induction to prove that any tree matching (aka generated by) grammar G has equal numbers of a's and b's. Use $A(T)$ and $B(T)$ as shorthand for the number of a's and b's in a tree T.

The induction variable is named _____ and it is the _____ of/in the tree.

Base Case(s):

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Inductive Step: