CS 173, Spring 18 Examlet 10, Part A

Name:____

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number $x > -1$, $(1+x)^n \geq 1 + nx$.

Let x be a real number with $x > -1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1+x)^n = (1+x)^0 = 1$ and $1 + nx = 1 + 0 = 1$. So $(1+x)^n \geq 1 + nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1+x)^n \geq 1 + nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1+x)^k \geq 1 + kx$. Notice that $(1+x)$ is positive since $x > -1$. So $(1+x)^{k+1} \geq (1+x)(1+kx)$.

But $(1+x)(1+kx) = 1 + x + kx + kx^2 = 1 + (1+k)x + kx^2$.

And $1 + (1+k)x + kx^2 \geq 1 + (1+k)x$ because kx^2 is non-negative.

So $(1+x)^{k+1} \geq (1+x)(1+kx) \geq 1 + (1+k)x$, and therefore $(1+x)^{k+1} \geq 1 + (1+k)x$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$

You may use the fact that $\sqrt{n+1} \geq \sqrt{n}$ for any natural number n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} = 1$ Also $\sqrt{n} = 1$. So $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p}} \geq \sqrt{n}$ for $n = 1, 2, \ldots, k$, for some integer $k \geq 1$.

Inductive Step: $\sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \sqrt{k}$ by the inductive hypothesis.

So

$$\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} = \frac{1}{\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p}} \geq \frac{1}{\sqrt{k+1}} + \sqrt{k} = \frac{1 + \sqrt{k}\sqrt{k+1}}{\sqrt{k+1}} \geq \frac{1 + \sqrt{k}\sqrt{k}}{\sqrt{k+1}} = \frac{1+k}{\sqrt{k+1}} = \sqrt{k+1}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p}} \geq \sqrt{k+1}$, which is what we needed to show.

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(15 points) Let function $f : \mathbb{Z}^+ \to \mathbb{R}$ be defined by

$f(1) = f(2) = 1$

$f(n) = \frac{1}{2}f(n-1) + \frac{1}{f(n-2)}$

Use (strong) induction to prove that $1 \leq f(n) \leq 2$ for all positive integers n.

Hint: prove both inequalities together using one induction.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$ and $n = 2$, $f(n) = 1$. So $1 \leq f(n) \leq 2$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $1 \leq f(n) \leq 2$ for $n = 1, 2, \ldots k-1$.

Inductive Step: From the inductive hypothesis, we know that $1 \leq f(k-1) \leq 2$ and $1 \leq f(k-2) \leq 2$.

So $\frac{1}{2} \leq \frac{1}{2}f(k-1) \leq \frac{1}{2} \cdot 2 = 1$ and $\frac{1}{2} \leq \frac{1}{f(k-2)} \leq \frac{1}{1} = 1$.

Using the upper bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \leq 1 + 1 = 2$.

Using the lower bounds from these equations: $f(k) = \frac{1}{2}f(k-1) + \frac{1}{f(k-2)} \geq \frac{1}{2} + \frac{1}{2} = 1$.

So $1 \leq f(k) \leq 2$, which is what we needed to show.

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(15 points) Recall the following fact about real numbers

Triangle Inequality: For any real numbers x and y, $|x+y| \leq |x| + |y|$.

Use this fact and (strong) induction to prove the following claim:

Claim: For any real numbers $x_1, x_2, \ldots, x_n$ ($n \geq 2$), $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, the claim is exactly the Triangle Inequality, which we're assuming to hold.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$ for any list of n real numbers $x_1, x_2, \ldots, x_n$, where $2 \leq n \leq k$.

Inductive Step: Let $x_1, x_2, \ldots, x_{k+1}$ be a list of $k+1$ real numbers.

Using the Triangle Inequality, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq |(x_1 + x_2 + \ldots + x_k)| + |x_{k+1}|$

But, by the inductive hypothesis $|x_1 + x_2 + \ldots + x_k| \leq |x_1| + |x_2| + \ldots + |x_k|$.

Putting these two equations together, we get

$|x_1 + x_2 + \ldots + x_k + x_{k+1}| = |(x_1 + x_2 + \ldots + x_k) + x_{k+1}| \leq (|x_1| + |x_2| + \ldots + |x_k|) + |x_{k+1}|$.

So $|x_1 + x_2 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + \ldots + |x_k| + |x_{k+1}|$, which is what we needed to show.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: $\prod_{p=1}^{n} \frac{2p-1}{2p} < \frac{1}{\sqrt{2n+1}}$ for all integers $n \geq 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n} \frac{2p-1}{2p} = \frac{1}{2} = \frac{1}{\sqrt{4}} < \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{2n+1}}$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n} \frac{2p-1}{2p} < \frac{1}{\sqrt{2n+1}}$ for $n = 1, \ldots, k$.

Inductive Step:

Notice that $(2k+1)(2k+3) = 4k^2 + 8k + 3 < 4k^2 + 8k + 4 = (2k+2)^2$.

So $\frac{2k+1}{(2k+2)^2} < \frac{1}{2k+3}$. So $\frac{(2k+1)^2}{(2k+2)^2} < \frac{2k+1}{2k+3}$.

Taking the square root of both sides gives us $\frac{2k+1}{2k+2} < \frac{\sqrt{2k+1}}{\sqrt{2k+3}}$. And therefore $\frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$.

Using this fact and the inductive hypothesis, we have

$$\prod_{p=1}^{k+1} \frac{2p-1}{2p} = \frac{2k+1}{2k+2}\left(\prod_{p=1}^{k} \frac{2p-1}{2p}\right) < \frac{2k+1}{2k+2}\frac{1}{\sqrt{2k+1}} < \frac{1}{\sqrt{2k+3}}$$

So $\prod_{p=1}^{k+1} \frac{2p-1}{2p} < \frac{1}{\sqrt{2k+3}}$, which is what we needed to show.

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(15 points) Suppose that $0 < q < \frac{1}{2}$. Use (strong) induction to prove the following claim:

Claim: $(1+q)^n \leq 1 + 2^n q$, for all positive integers n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(1+q)^n = 1+q$ Also $1 + 2^n q = 1 + 2q$. So $(1+q)^n \leq 1 + 2^n q$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(1+q)^n \leq 1+2^n q$, for $n = 1, 2, \ldots, k$.

Inductive Step: From the inductive hypothesis, we know that $(1+q)^k \leq 1 + 2^k q$.

At $n = k+1$, we have

$$\begin{aligned} (1+q)^{k+1} &= (1+q)(1+q)^k \leq (1+q)(1+2^k q) \\ &= 1 + q + 2^k q + 2^k q^2 = 1 + q(1 + 2^k + 2^k q) \end{aligned}$$

Recall that $q < \frac{1}{2}$, so $2^k q < 2^{k-1}$. Also notice that $1 \leq 2^{k-1}$. Using these facts, we get

$$(1+q)^{k+1} \quad \leq \quad = 1 + q(1 + 2^k + 2^k q) \leq 1 + q(2^{k-1} + 2^k + 2^{k-1}) = 1 + 2^{k+1} q$$

So $(1+q)^{k+1} \leq 1 + 2^{k+1} q$, which is what we needed to show.