CS 173, Spring 18 Examlet 10, Part B

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 4.

   $T(1) = 7$ $\qquad$ $T(n) = 2T\left(\frac{n}{4}\right) + n$

   (a) The height: $\log_4 n$

   (b) Number of leaves: $2^{\log_4 n} = n^{1/2} = \sqrt{n}$
   [Ok to stop simplifying at $n^{1/2}$.]

   (c) Total work (sum of the nodes) at level k (please simplify):
   There are 2^k nodes at level k. Each of these nodes contains the value $n/4^k$. So the total work is $2^k \cdot n/4^k = n/2^k$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $42n!$ $\quad$ 7^n $\quad$ $100\log n$ $\quad$ $n\log(n^7)$ $\quad$ 2^{3n} $\quad$ $\log(2^n)$ $\quad$ $(n^3)^7$

   Solution:

   $100\log n \ll \log(2^n) \ll n\log(n^7) \ll (n^3)^7 \ll 7^n \ll 2^{3n} \ll 42n!$

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1. (7 points) In class, Prof. Snape made the following claim about all functions g and h from the reals to the reals whose output values are always > 1. If $g(x) \ll h(x)$, then $\log(g(x)) \ll \log(h(x))$. Is this true? Briefly justify your answer.

   Solution:

   This is not true. Consider $f(x) = x$ and $g(x) = x^2$. Then $\log(g(x)) = 2\log(f(x))$ So it can't be the case that $\log(f(x)) \ll \log(g(x))$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$ $T(n) = T(n/2) + c$	$\Theta(\log n)$	☑	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐
$T(1) = d$ $T(n) = 2T(n-1) + c$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☑	$\Theta(3^n)$	☐
2^n is	$\Theta(3^n)$	☐	$O(3^n)$	☑	neither of these	☐		
Suppose f and g produce only positive outputs and $f(n) \ll g(n)$. Will $f(n)$ be $\Theta(g(n))$?	no	☑	perhaps	☐	yes	☐		

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1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that f is $O(g)$ and g is $O(h)$. Must f be $O(h)$? Briefly justify your answer.

   Solution: This is true. Since f is $O(g)$ and g is $O(h)$, there are positive reals c, k, C and K such that $0 \leq f(x) \leq cg(x)$ and $0 \leq g(y) \leq Ch(y)$ for every $x \geq k$ and $y \geq K$.

   But then if we let $p = cC$, we have $O \leq f(x) \leq ph(x)$ for every $x \geq \max(k, K)$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$ $T(n) = T(n/2) + n$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☑ √	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐
$T(1) = d$ $T(n) = 3T(n-1) + c$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☑ √

$n^{log_2 5}$ grows: faster than n^2 ☑ √ slower than n^2 ☐ at the same rate as n^2 ☐

Suppose $f(n)$ is $O(g(n))$. Will $g(n)$ be $O(f(n))$? no ☐ perhaps ☑ √ yes ☐

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NetID:_________________________ **Lecture:** **A** **B**

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1. (7 points) Suppose that f, g, and h are functions from the reals to the reals, such that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Must $f(x)g(x)$ be $O(h(x)h(x))$?

   Solution: This is true. Since $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$, there are positive reals c, k, C and K such that $0 \leq f(x) \leq ch(x)$ and $0 \leq g(y) \leq Ch(y)$ for every $x \geq k$ and $y \geq K$.

   But then if we let $p = cC$, we have $O \leq f(x) \leq ph(x)h(x)$ for every $x \geq \max(k, K)$.

2. (8 points) Check the (single) box that best characterizes each item.

Suppose $f(n)$ is $O(g(n))$. Will $f(n)$ be $\Theta(g(n))$?	no	☐	perhaps	☑	yes	☐		
$17n^3$	$\Theta(n^3)$	☑	$O(n^3)$	☐	neither of these	☐		
$T(1) = c$, $T(n) = 2T(n/2) + n^2$	$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☑	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐
$T(1) = d$, $T(n) = T(n/3) + c$	$\Theta(\log n)$	☑	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
	$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

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1. (9 points) Fill in key facts about the recursion tree for T, assuming that n is a power of 2.

   $T(1) = 1 \qquad T(n) = 4T\left(\frac{n}{2}\right) + n^2$

   (a) Value in each node at level k: $\left(\frac{n}{2^k}\right)^2 = \frac{n^2}{4^k}$

   (b) Total work (sum of the nodes) at level k (please simplify): Level k has 4^k nodes, each containing the value $\frac{n^2}{4^k}$. So the total for the level is $4^k\frac{n^2}{4^k} = n^2$

   (c) Sum of the work in all internal (non-leaf) nodes (please simplify):
   The number of non-leaf levels is the height of the tree, which is $\log n$. The work at each level is n^2. So the total work in all the non-leaf nodes is $n^2 \log n$.

   Change of base formula: $\log_b n = (\log_a n)(\log_b a)$

2. (6 points) Write the following functions in the boxes so that f is to the left of g if and only if $f(n) \ll g(n)$.

   $n \log n \qquad \log(n^{17}) \qquad \sqrt{n} + n! + 18 \qquad 2^n \qquad 8n^2 \qquad 8^{\log_8 n} \qquad 0.001n^3$

   Solution:

   $\log(n^{17}) \ll 8^{\log_8 n} \ll n \log n \ll 8n^2 \ll 0.001n^3 \ll 2^n \ll \sqrt{n} + n! + 18$

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1. (7 points) Suppose that f and g are functions from the reals to the reals. Define precisely what it means for f to be $O(g)$.

   Solution: There are positive reals c and k such that $0 \leq f(x) \leq cg(x)$ for every $x \geq k$.

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = c$
$T(n) = 3T(n/3) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	√
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = 2T(n/2) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	√	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

2^n — $\Theta(n!)$ ☐ $O(n!)$ √ neither of these ☐

Suppose f and g produce only positive outputs and $f(n) \ll g(n)$. Will $f(n)$ be $O(g(n))$? — no ☐ perhaps ☐ yes √