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(8 points) The Emerald City bakery allows customers to special-order fruit pies. Each pie can contain one fruit or a mixture of 2 or 3 (distinct) fruits. The available fruits are raspberry, blueberry, pear, apple, cherry, apricot, and peach. How many choices do you have?

Solution:

There are 7 choices for the 1-fruit pies, $\binom{7}{2}$ for the 2-fruit pies, and $\binom{7}{3}$ for the 3-fruit pies. So the total number of choices is $7 + \binom{7}{2} + \binom{7}{3}$. If you want to turn this into a single number (not required), it is $7 + 21 + 35 = 63$.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a dorm room d, such that d has green walls and d has no window.

Solution: For every dorm room d, d has walls that aren't green or d has a window.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select an ordered sequence of 17 flowers chosen from 4 possible varieties.	$\binom{16}{3}$ ☐	$\binom{16}{4}$ ☐	$\binom{20}{3}$ ☐
	$\binom{20}{4}$ ☐	$\binom{21}{3}$ ☐	4^{17} ☑

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(9 points) Use proof by contradiction to show that there are no positive integer solutions to the equation $4x^2 - y^2 = 1$.

Solution: Suppose not. That is, suppose that there are positive integers x and y such that $4x^2 - y^2 = 1$. Factoring the lefthand side, we get $(2x - y)(2x + y) = 1$. $(2x - y)$ and $(2x + y)$ must be integers since x and y are integers. So $(2x - y)$ and $(2x + y)$ are either both 1 or both -1.

Case 1: $(2x - y) = 1$ and $(2x + y) = 1$. Adding the two equations gives us $4x = 2$, so $x = 1/2$.

Case 2: $(2x - y) = -1$ and $(2x + y) = -1$. Adding the two equations gives us $4x = -2$, so $x = -1/2$.

In both cases, x must have a non-integer value, contradicting our assumption that x is an integer.

(6 points) In the game Tic-tac-toe is played on a 3x3 grid and a move consists of the first player putting an X into one of the squares, or the second player putting an O into one of the squares. The board cannot be rotated, e.g. an X in the upper right corner is different from an X in the lower left corner. How many different board configurations are possible after four moves (i.e. two moves by each player)?

Solution: You need to pick 2 of the 9 squares to contain the X's, and then 2 of the remaining 7 squares to contain the O's. So the total number of choices is $\binom{9}{2}\binom{7}{2}$.

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(8 points) Anna has to climb 6 stairs to get onto the podium. She can climb a single step or two steps at a time. E.g. one possible (ordered) sequence of actions is: one step, a double step, three single steps. How many ways could she climb the stairs?

Solution: There is one way to do it with only single steps. Likewise, there is only one way to do it with three double-steps.

There are 5 ways to order one double step plus four single steps. And there are $\binom{4}{2} = 6$ ways to order two double steps plus two single steps.

So the total number of options is $1 + 1 + 5 + 6 = 13$.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

> There is a bug b, such that for every plant p, if b pollinates p and p is showy, then p is poisonous.

Solution: For every bug b, there is a plant p, such that b pollinates p and p is showy, but p is not poisonous.

(2 points) Check the (single) box that best characterizes each item.

The number of ways to select a set of 17 flowers chosen from 4 possible varieties (zero or more of each variety).	$\binom{17}{5}$ ☐	$\binom{20}{4}$ ☐	$\binom{20}{3}$ ☑
	$\binom{17}{4}$ ☐	$\binom{21}{4}$ ☐	$\frac{17!}{4!}$ ☐

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(8 points) Ignatius Eggbert flips a coin 10 times. The coin is fair, i.e. equal chance of getting a head vs. a tail. What is the chance that he gets exactly 7 heads? Give an exact formula; don't try to figure out the decimal equivalent. Briefly explain your answer and/or show work.

Solution: There are 2^{10} sequences of 10 coin flips. There are $\binom{10}{7}$ ways to pick 7 of these flips to contain a head. So the chance of getting 7 heads is

$$\frac{\binom{10}{7}}{2^{10}}$$

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

For every Meerkat m, if m is in New York, then m is not in the wild or m is lost.

Solution: There is a Meerkat m, such that m is in New York, but m is in the wild and m is not lost.

(2 points) Check the (single) box that best characterizes each item.

The number of bit strings of length 20 with exactly 7 1's.	$\binom{26}{7}$ ☐	$\binom{27}{7}$ ☐	$\binom{26}{6}$ ☐
	$\binom{20}{13}$ ☑	$\binom{20}{14}$ ☐	2^{20} ☐

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(9 points) Use proof by contradiction to show that $\sqrt{2} + \sqrt{3} \leq 4$.

Solution: Suppose not. That is, suppose that $\sqrt{2} + \sqrt{3} > 4$.

Then $(\sqrt{2} + \sqrt{3})^2 > 16$. (All the numbers involved are positive.) So $2 + 2\sqrt{2}\sqrt{3} + 3 > 16$. So $2\sqrt{2}\sqrt{3} > 11$.

Squaring both sides again, we get $4 \cdot 2 \cdot 3 > 121$. That is $24 > 121$. But this last equation is obviously false. So our original assumption must have been wrong and therefore $\sqrt{2} + \sqrt{3} \leq 4$.

(6 points) Use the binomial theorem to find a closed form for the summation $\sum_{k=0}^{n}(-1)^k\binom{n}{k}$. Make sure it's clear how you used the theorem.

Solution: The binomial theorem states that $(x+y)^n = \sum_{k=0}^{n} x^k y^{n-k}\binom{n}{k}$.

Setting $x = -1$ and $y = 1$ gives us $(-1+1)^n = \sum_{k=0}^{n}(-1)^k 1^{n-k}\binom{n}{k}$

That is $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.

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(8 points) If w is a string of characters, a *substring* of w is a contiguous string that forms part of w. For example, "rtho" is a substring of "warthog" but "ahog" is not. Suppose that w is a string without any repeated characters, of length k. How many non-empty substrings does it have?

Solution: A substring is determined by its starting and ending positions. This is an unordered pair, because we only get a well-defined substring if the starting position precedes the ending one. (Or, if you like, the backwards pair defines the same substring.)

Method (1): Suppose a "position" is a character in the string. There are k substrings of length 1. For substrings of length $k > 1$, we need to pick a set of two distinct positions (starting and ending). We have $\binom{k}{2}$ ways to do this. Adding these two numbers gives us $\frac{k(k-1)}{2} + k = \frac{k(k+1)}{2}$.

Method (2): Suppose a "position" lies between two characters. Then we have $k+1$ positions, from which we must choose an unordered pair. We have $\binom{k+1}{2} = \frac{(k+1)k}{2}$ ways to make this choice.

(5 points) State the negation of the following claim, moving all negations (e.g. "not") so that they are on individual predicates.

There is a soup s such that s is tasty and s does not contain meat.

Solution: For every soup s, s is not tasty or s contains meat.

(2 points) Check the (single) box that best characterizes each item.

How many ways can I choose 6 bagels from among 8 varieties, if I can have any number of bagels from any type?

$\frac{8!}{6!2!}$ ☐	$\frac{13!}{6!7!}$ ☑	$\frac{14!}{9!5!}$ ☐
$\frac{14!}{6!7!}$ ☐	8^6 ☐	6^8 ☐