# CS 173, Summer 19 Examlet 8

**Name:**________________________________________

**NetID:**________________________ **Lecture:** **A**

**Discussion:** **Monday & Wednesday** **1:30** **2:30**

(20 points) Recall that $F_n$ is the nth Fibonacci number, and these start with $F_0 = 0$, $F_1 = 1$. Use (strong) induction to prove the following claim:

Claim: $F_n < (5/3)^n$ for any natural number $n$.

**Solution:**

Proof by induction on $n$.

**Base Case(s):** At $n = 0$, $F_n = 0$ and $(5/3)^n = 1$, so $F_n < (5/3)^n$.

At $n = 1$, $F_n = 1$ and $(5/3)^n = 5/3$, so $F_n < (5/3)^n$.

**Inductive Hypothesis** [Be specific, don't just refer to "the claim"]: Suppose that $F_n < (5/3)^n$, for $n = 0, 1, \ldots, k$.

**Inductive Step:** By the inductive hypothesis, $F_k < (5/3)^k$ and $F_{k-1} < (5/3)^{k-1}$. So

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
&< (5/3)^k + (5/3)^{k-1} \\
&= (5/3+1)(5/3)^{k-1} = (8/3)(5/3)^{k-1} = (24/9)(5/3)^{k-1} \\
&< (25/9)(5/3)^{k-1} = (5/3)^{k+1}
\end{aligned}$$

So $F_{k+1} < (5/3)^{k+1}$, which is what we needed to prove.

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(10 points) Suppose we have a function $F$ defined (for $n$ a power of 2) by

$$\begin{aligned} F(2) &= c \\ F(n) &= F(n/2) + n \text{ for } n \geq 4 \end{aligned}$$

Your partner has already figured out that

$$F(n) = F(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i}$$

Finish finding the closed form for $F$. Show your work and simplify your answer.

**Solution:**

To find the value of $k$ at the base case, we need to set $n/2^k = 2$. This means that $n = 2 \cdot 2^k$. So $n = 2^{k+1}$. So $k + 1 = \log n$. So $k = \log n - 1$. Substituting this value into the above equation, we get

$$\begin{aligned} F(n) &= T(n/2^k) + \sum_{i=0}^{k-1} n\frac{1}{2^i} = F(2) + \sum_{i=0}^{\log n - 2} n\frac{1}{2^i} \\ &= c + n\sum_{i=0}^{\log n - 2} \frac{1}{2^i} = c + n(2 - \frac{1}{2^{\log n - 2}}) \\ &= c + n(2 - \frac{1}{2^{\log n} \cdot 2^{-2}}) = c + n(2 - \frac{4}{2^{\log n}}) \\ &= c + n(2 - \frac{4}{n}) = c + 2n - 4 \end{aligned}$$