CS 173, Summer 19 Examlet 10

Name:____

NetID:____________________ **Lecture: A**

Discussion: Monday & Wednesday 1:30 2:30

(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} < 4^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 < 16 = 4^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} < 4^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 4^k$.

Then we can compute

$$\begin{aligned}
\frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\
&< \frac{(2k+2)(2k+1)}{(k+1)^2}4^k \\
&< \frac{(2k+2)(2k+2)}{(k+1)^2}4^k = \frac{4(k+1)(k+1)}{(k+1)^2}4^k \\
&= 4 \cdot 4^k = 4^{k+1}
\end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} < 4^{k+1}$, which is what we needed to show.

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1. (7 points) You found the following claim on a hallway whiteboard. Suppose that f and g are increasing functions from the reals to the reals, for which all output values are > 1. If $f(x)$ is $O(g(x))$, then $\log(f(x))$ is $O(\log(g(x)))$. Is this true? Briefly justify your answer.

   Solution:

   Yes, it is true. Suppose that $f(x)$ is $O(g(x))$. Then there are positive reals c and k such that $f(x) \leq cg(x)$ for all $x \geq k$. Then $\log(f(x)) \leq \log c + \log(g(x))$ for all $x \geq k$. Since $g(x)$ is an increasing function and c isn't, there is some $K \geq k$ such that $\log c \leq \log(g(x))$. So then $\log(f(x)) \leq 2\log(g(x))$ for all $x \geq K$. So $\log(f(x))$ is $O(\log(g(x)))$.

   [You don't need this much technical detail for full credit. We can ignore the other inequality from the definition of big-O because the two functions always output positive values.]

2. (8 points) Check the (single) box that best characterizes each item.

$T(1) = d$
$T(n) = 2T(n-1) + c$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	☐	$\Theta(n^3)$	☐	$\Theta(2^n)$	√	$\Theta(3^n)$	☐

$T(1) = d$
$T(n) = T(n-1) + n$

$\Theta(\log n)$	☐	$\Theta(\sqrt{n})$	☐	$\Theta(n)$	☐	$\Theta(n \log n)$	☐
$\Theta(n^2)$	√	$\Theta(n^3)$	☐	$\Theta(2^n)$	☐	$\Theta(3^n)$	☐

$n^{1.5}$ is

$\Theta(n^{1.414})$	☐	$O(n^{1.414})$	☐	neither of these	√

Suppose $f(n)$ is $\Theta(g(n))$.
Will $g(n)$ be $O(f(n))$?

no	☐	sometimes	☐	yes	√