University of Illinois Spring 2020

ECE 313: Midterm Exam I (Conflict)

Wednesday, March 05, 2020

1. **[20 points]**

   (a) **[10 points]** You are playing a game that involves rolling two six-sided dice. You roll the two dice once and have a choice to either keep the highest value rolled, or you can roll again and keep the average of the two dice values of the second roll. Your goal is to obtain the highest expected score.

   For what values of the first roll should you choose to roll again?
   Solution: If you roll two dice, their average is simply the average of a single die, which is 3.5. Hence the best strategy is to roll again whenever the maximum value of the first roll is ≤ 3.

   (b) **[10 points]** The scoring of the game changes. You roll once and have a choice to either keep the highest value squared, or you can roll again and keep the product of the two dice values on the second. For instance, if you first roll $(3, 5)$ you can either keep 25, or you can roll again and take the product of the second roll. Your goal is to obtain the highest expected score.

   For what values of the first roll should you choose to roll again?
   Solution: We now need to compute the expected product when rolling two dice. A quick calculation shows this value to be

$$\frac{1}{36}\Big(1 + 2 \cdot (2 + 3 + 4 + 5 + 6) + 4 + 2 \cdot (6 + 8 + 10 + 12) + 9 + 2 \cdot (12 + 15 + 18) + 16 + 2 \cdot (20 + 24) + 25 + 2 \cdot 30 + 36\Big) = \frac{441}{36} = \frac{49}{4} = 12.25. \tag{1}$$

   This is greater than $3^2 = 9$ and smaller than $4^2 = 16$. Hence the optimal strategy is to roll again if the maximum of the first roll is smaller than 4.

2. **[20 points]** There are two dice in a bag. One is a standard die with faces 1,2,3,4,5 and 6. The other one is a non-standard die with faces 2,2,4,6,6,6.

   (a) **[10 points]** One die is selected at random from the bag and rolled. What is the probability that the number shown is less than or equal to 3?
   Solution: Let E_1 be the event that the standard die is chosen and E_2 the event that the non-standard die is chosen. Let A be the event that we observe a number less than or equal to 3. By the law of total probability,

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{12}.$$

   (b) **[10 points]** The randomly selected die is rolled three times and the numbers shown are 2,2, and 4 (in this order). What is the conditional probability that the standard die was chosen?
   Solution: Let B be the event that we observe 2,2, and 4. By Bayes' rule,

$$\begin{aligned} P(E_1|B) &= \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2)} \\ &= \frac{(1/6)^3(1/2)}{(1/6)^3(1/2) + (1/3)^2(1/6)(1/2)} = \frac{1/36}{1/36 + 1/9} = \frac{1}{5}. \end{aligned}$$

3. **[20 points]**

   (a) **[6 points]** Suppose 6 letters {B,A,N,A,N,A} are placed in a hat and shuffled. You randomly choose one letter at a time without replacement. What is the probability that you will spell the word BANANA in the order you draw the letters? Assume that each letter has an equal probability of being drawn.

   Solution: In total there are 6! sequences that can be drawn with equal probability. If we assume the letters are labeled, then one such sequence will be $B_1A_2N_3A_4N_5A_6$. If we now remove the labels, this will allow for $3! \cdot 2!$ sequences that lead to the same spelling of BANANA, where we are accounting for a multiplicity of 3, and 2 on the letters A and N, respectively. Hence, the probability of choosing one sequence spelling BANANA is

$$\frac{3! \cdot 2!}{6!} = \frac{1}{5 \cdot 4 \cdot 3} = \frac{1}{60}. \tag{2}$$

   (b) **[14 points]** Suppose again your goal is to spell BANANA by drawing the correct letters in the correct order, but now you draw with replacement on each letter until you get the right one. Once you get the correct letter, you do not replace it in the hat. What is the expected number of draws from the hat you must perform to correctly spell BANANA? Note, you must spell it in the right order! For instance, you must first draw a B before you can move on to draw for an A, etc.

   Solution: Let N_i be the number of draws taken to get the i^{th} letter in BANANA. Since we assume each letter has an equal probability of being drawn, the probability distribution $p_{N_i}(k)$ is a geometric distribution. Namely, $p_{N_i}(k) = (1-p_i)^{k-1}p_i$, where p_i is the probability of drawing the correct letter in the i^{th} slot. Since $E[N_i] = \frac{1}{p_i}$, we then compute

$$\begin{aligned} &E[N_1] = 6, && E[N_2] = \frac{5}{3}, && E[N_3] = \frac{4}{2} \\ &E[N_4] = \frac{3}{2}, && E[N_5] = \frac{2}{1}, && E[N_6] = \frac{1}{1} \end{aligned} \tag{3}$$

   Therefore,

$$E[N_{tot}] = \sum_{i=1}^{6} E[N_i] = 6 + \frac{5}{3} + \frac{4}{2} + \frac{3}{2} + \frac{2}{1} + 1 = \frac{85}{6}$$

4. **[20 points]** I have two coins in my pocket. One is a fair coin and the other is a biased coin with with $P\{\text{heads}\} = p$. I randomly select one of the coins, and toss it 10 times. Let X be the total number of heads observed.

   (a) **[10 points]** Compute the pmf of X as a function of p.

   Solution: Let E_1 be the event that I choose the fair coin, and E_2 the event that I choose the biased coin. For $k = 0, 1, ..., 10$, using the law of total probability we have

$$\begin{aligned} p_X(k) = P\{X = k\} &= P(X = k|E_1)P(E_1) + P(X = k|E_2)P(E_2) \\ &= \frac{1}{2}\left(\binom{10}{k}(1/2)^{10} + \binom{10}{k}p^k(1-p)^{10-k}\right) \\ &= \frac{1}{2}\binom{10}{k}\left((1/2)^{10} + p^k(1-p)^{10-k}\right) \end{aligned}$$

   For $k > 0$ or $k > 10$, the pmf is just $p_X(k) = 0$.

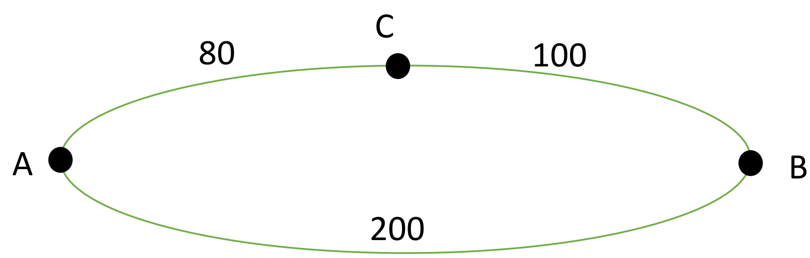
(b) **[10 points]** Given that $X = 7$, what is the maximum likelihood estimate of p?
Solution: To find the ML estimate of p, we can take the derivative of $p_X(7)$ with respect to p and set it to 0. Notice that the term $\frac{1}{2}\binom{10}{k}$ is just a constant and the term $(1/2)^{10}$ does not depend on p. Hence, taking the derivative of $p_X(7)$ and setting it to zero is equivalent to taking the derivative of just $p^7(1-p)^3$ and setting it to zero. Thus

$$\frac{d}{dp}p^7(1-p)^3 = 7p^6(1-p)^3 + 3(1-p)^2p^7(-1) = p^6(1-p)^2\,(7(1-p) - 3p) = 0,$$

which implies that

$$7(1-p) = 3p \;\Rightarrow\; \hat{p}_{ML} = \frac{7}{10}.$$

5. **[20 points]** Messages can be transmitted between City A and City B either directly or via city C on communication links whose capacities are indicated on the diagram shown below. Assume that the three links fail independently with equal probability $p = 0.1$. Let X denote the communication capacity between City A and City B.



(a) **[10 points]** Find the probability of outage for communication between cities A and B. This is the probability that A is unable to communicate with B via any link.
Solution: The communication network fails if both the direct link AB and the path ACB fail. $P(\text{Direct Link AB fails}) = p$, and $P(\text{Path ACB fail}) = P(\text{AC fails} \cup \text{CB fails}) = p + p - p^2 = 2p - p^2$

$$P(\text{network fail}) = p \times (2p - p^2) = 2p^2 - p^3 = 0.019 = \frac{19}{1000}.$$

(b) **[10 points]** What values can X take on? What is the distribution of X?
Solution: X can take values of 0, 80, 200, and 280. The probability that the path ACB fails is $p + p - p^2 = 2p - p^2$.

$$P(X = 0) = p \times (p + p - p^2) = 2p^2 - p^3 = \frac{19}{1000}$$
$$P(X = 80) = p \times (1-p) \times (1-p) = p - 2p^2 + p^3 = 0.1 - 0.02 + 0.001 = \frac{81}{1000}$$
$$P(X = 200) = (1-p) \times (p + p - p^2) = 2p - 3p^2 + p^3 = 0.2 - 0.03 + 0.001 = \frac{171}{1000}$$
$$P(X = 280) = (1-p)^3 = 1 - 3p + 3p^2 - p^3 = 1 - 0.3 + 0.03 - 0.001 = \frac{729}{1000}$$