University of Illinois Fall 2022

ECE 313: Midterm Exam I

Monday, October 03, 2022

1. **[12 points]** In a game of dice, each contestant picks a four-sided fair die and throws it. Whoever gets the larger number wins the game. Alice and Bob are playing this game of dice. There are three dice available for them to pick:

   - Die 1: a fair four-sided die with numbers 1, 5, 7 and 12 on it.
   - Die 2: a fair four-sided die with numbers 2, 6, 8 and 10 on it.
   - Die 3: a fair four-sided die with numbers 3, 4, 9 and 11 on it.

   Bob picks a die first, and Alice chooses one of the remaining dice.

   (a) **[4 points]** Let Ω denote the sample space of this game. The elements of Ω are of the form (B, A), where B is the number Bob gets after rolling his die, and A is the number that Alice gets after rolling her die. How many elements does Ω have?

   Solution: There are 12 possible numbers for the first roll, and 8 possible numbers for the second roll. Hence, $|\Omega| = 12 \times 8$

   (b) **[8 points]** Suppose that Bob picks Die 2. Is for Alice better to choose Die 1 or Die 3? Explain your answer.

   Solution: If Alice picks Die 1, she wins if $B = 2$ and $A \in \{5, 7, 12\}$, or $B = 6$ and $A \in \{7, 12\}$, or $B = 8$ and $A = 12$, or $B = 10$ and $A = 12$; hence, in this case her probability of winning is 7/16. If Alice picks Die 3, she wins if the outcome of the rolls are $B = 2$ and $A \in \{3, 4, 9, 11\}$, or $B = 6$ and $A \in \{9, 11\}$, or $B = 8$ and $A = \{9, 11\}$, or $B = 10$ and $A = 11$, hence, in this case her probability of winning is 9/16. Her best strategy is to choose Die 3.

2. **[20 points]** The following parts are separate.

   (a) **[5 points]** Suppose X is a Poisson random variable with mean 8. What is $\mathrm{Var}(3X+4)$?

   Solution: The parameter λ for the Poisson distribution is $\lambda = 8$. $\mathrm{Var}(X) = \lambda = 8$. Use scaling rule $\mathrm{Var}(3X + 4) = 9\mathrm{Var}(X) = 72$.

   (b) **[5 points]** Find $\mathbb{E}[5X^2 + X - 1]$ for a geometric random variable with mean 5.

   Solution: For a geometric random variable X, $\mathbb{E}[X] = \frac{1}{p}$, so $p = 1/\mathbb{E}[X] = \frac{1}{5}$. $\mathbb{E}[5X^2 + X - 1] = 5\mathbb{E}[X^2] + \mathbb{E}[X] - 1 = 5(\mathrm{Var}(X) + \mathbb{E}[X]^2) + \mathbb{E}[X] - 1 = 5(\frac{1-p}{p^2} + \frac{1}{p^2}) + \frac{1}{p} - 1 = 5 \times (25 \times (1 - 1/5) + 25) + 4 = 229$.

   (c) **[5 points]** If Z denotes the number of occurrences of an event of probability $p = \frac{1}{3}$ on 15 independent trials, what is the conditional expected value of Z given that the event occurred 4 times on the first 10 trials?

   Solution: What happens on the last 5 trials is independent of what happens on the first 10. Thus, the number of occurrences of the event on the last 5 trials is a binomial random variable with parameters $(n = 5, p)$, and hence expected value $5p$. This is true regardless of what happened on the first 5 trials. Therefore the conditional expectation of Z is $4 + 5p$.

   (d) **[5 points]** Roll 4 fair dice. Let X be the total number of 6's. Find $P(X \geq 2|X \geq 1)$.

   Solution: The probability of getting no 6s on the 4 dice is $p_0 = \left(\frac{5}{6}\right)^4$ and the probability of getting one 6 is $p_1 = \binom{4}{1}\left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)$. Therefore, the conditional probability we seek is $\frac{1-p_0-p_1}{1-p_0}$.

3. **[10 points]** Let X, Y be two independent Bernoulli random variables such that $P(X=1)=\frac{1}{4}$ and $P(Y=1)=\frac{3}{4}$. Define the random variables $Z = X+Y$ and $W = |X-Y|$.

   True or False: Z, W are independent random variables. Justify your answer.

   Solution: False:

   $P(Z=0,W=0) = P(X=0,Y=0) = P(X=0)P(Y=0) = \frac{3}{4}\frac{1}{4} = \frac{3}{16}$, while

   $P(Z=0)P(W=0) = P(X=0,Y=0)(P(X=0,Y=0)+P(X=1,Y=1)) = \frac{3}{16}(\frac{3}{16}+\frac{3}{16}) \neq \frac{3}{16}$.

4. **[16 points]** Suppose that in the first round of a game, 4 fair dice are rolled independently. In the second round of this game, only the dice with outcome 6 in the first round are rolled again independently. Let X be the number of dice with outcome 6 in the second round. Find the pmf of X.

   Solution: The number of dice with outcome 6 in the second round is the same as if we roll the dice twice (i.e., for two rounds) and count the number of dice with outcome 6 in both rounds. Clearly, each die has twice outcome 6 with probability $\frac{1}{6^2} = \frac{1}{36}$ and therefore, $X \sim \text{Bin}(4, 1/36)$.

5. **[20 points]** A Bungee jumping company, wants to ensure the quality of the ropes that it uses for each jump. A rope fails a test with probability p.

   (a) **[10 points]** The company tested n ropes, and X of them failed. Considering the point estimator $\widehat{p} = X/n$, how large should n be so that the company is 96% confident that p is within 0.05 of $\widehat{p}$?
   Solution:

   $$1 - \frac{1}{a^2} = 96\% \longrightarrow a = 5 \tag{1}$$
   $$\frac{a}{2\sqrt{n}} = 0.05 \longrightarrow n \geq 2500 \tag{2}$$

   (b) **[10 points]** Suppose that the company uses 2 ropes for each jump to decrease the chance of failure. A jump results in injury if both the 2 ropes fail. Suppose that each rope fails with probability p independently of each other. Among 100 jumps so far, only 2 of them resulted in injury. What is the maximum likelihood estimate of p?
   Solution: Let X denote the number of jumps that resulted in an injury, notice that X has a Binomial distribution with parameters $(100, p^2)$. The probability of observing 2 injuries out of 100 jumps is given by

   $$P(X=2) = \binom{100}{2}\left(p^2\right)^2\left(1-p^2\right)^{98} \tag{3}$$

   To maximize the right-hand side, we set the derivative to be equal to zero.

   $$\frac{\mathrm{d}P(X=2)}{\mathrm{d}p} = 0 \longrightarrow 4p^3\left(1-p^2\right)^{98} - 98 \times 2p \times p^4\left(1-p^2\right)^{97} = 0 \longrightarrow 2 - 2p^2 = 98p^2 \tag{4}$$

   Hence, $\widehat{p}_{\text{ML}} = \sqrt{0.02}$.

6. **[22 points]** Consider a binary hypothesis testing problem, where X has the uniform distribution on $k = 1, 2, \ldots, 6$ under hypothesis H_0 and Geometric(1/2) under hypothesis H_1.

(a) **[8 points]** Describe the ML decision rule. Express it in a simplified form.

Solution: Under hypothesis H_1, $p_1(k) = \left(\frac{1}{2}\right)^k$ for integer k. The likelihood under H_0 is $p_0(k) = \frac{1}{6}$ for $k \in \{1, 2, \ldots, 6\}$ and zero else. For $k > 6$, the ML rule chooses the largest one of the two, which is always $p_1(k) > p_0(k)$ for $k > 6$ because $p_0(k) = 0$ there. For $k \in \{1, 2, \ldots, 6\}$, $p_0(k) = \frac{1}{6} > p_1(k) = \left(\frac{1}{2}\right)^k$ is not true for $k \in \{1, 2\}$, but it is true for $k \in \{3, 4, 5, 6\}$. Hence, the ML rule declares H_0 if $X \in \{3, 4, 5, 6\}$, and H_1 else (if $X \in \{1, 2, 7, 8, \ldots\}$).

(b) **[8 points]** Describe the MAP decision rule under the assumption that H_0 is a priori three times as likely as H_1. Express it in a simplified form.

Solution: MAP rule chooses H_1 if the likelihood ratio $\Lambda(k) = \frac{p_1(k)}{p_0(k)} > \frac{\pi_0}{\pi_1} = 3$. For $k > 6$, since $p_0(k) = 0$, MAP rule chooses H_1. For $k \in \{1, 2, \ldots, 6\}$, $\Lambda(k) = 6(1/2)^k > 3$ if $k > 1$. Therefore, the MAP rule declares H_0 if $X = \{2, 3, 4, 5, 6\}$, and H_1 else if $X > 6$). For $k = 1$, we have $\Lambda(k) = 6(1/2) = 3$, and MAP can declares either H_0 or H_1.

(c) **[6 points]** Find the average error probability, p_e, for the MAP rule, using the same prior distribution given in part (b).

Solution: Since $\pi_0 = 3\pi_1$ and $\pi_0 + \pi_1 = 1$, we get $\pi_0 = \frac{3}{4}$ and $\pi_1 = \frac{1}{4}$.
Under the MAP decision rule, assuming it declares H_1 for $X = 1$, we have

$$p_{\text{false alarm}} = \sum_{k:\text{declare } H_1} p_0(k) = \frac{1}{6}$$

$$p_{\text{miss}} = \sum_{k:\text{declare } H_0} p_1(k) = \sum_{k=2}^{6} \left(\frac{1}{2}\right)^k = \frac{31}{64}$$

$$p_{e,\text{MAP}} = \pi_0 \times p_{\text{false alarm}} + \pi_1 \times p_{\text{miss}} = \frac{3}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{31}{64} = \frac{63}{256},$$

and if we assume the MAP decision rule declares H_1 for all $X > 7$, we have

$$p_{\text{false alarm}} = \sum_{k:\text{declare } H_1} p_0(k) = 0$$

$$p_{\text{miss}} = \sum_{k:\text{declare } H_0} p_1(k) = \sum_{k=1}^{6} \left(\frac{1}{2}\right)^k = \frac{63}{64}$$

$$p_{e,\text{MAP}} = \pi_0 \times p_{\text{false alarm}} + \pi_1 \times p_{\text{miss}} = \frac{3}{4} \times 0 + \frac{1}{4} \times \frac{63}{64} = \frac{63}{256}.$$

Since the MAP minimizes p_{error}, and both H_0 and H_1 are acceptable for $k = 1$, the resulted $p_{e,\text{MAP}}$ is independent of the decision rule for $k = 1$.