University of Illinois Fall 2022

ECE 313: Midterm Exam II

Monday, Nov 07, 2022

1. **[12 points]** Let X be a nonnegative continuous random variable such that $P(X > x+y|X > y) = P(X > x)$ for all $x \geq 0$ and all $y \geq 0$. Let n be an arbitrary positive integer. Show that $P(X > n) = (P(X > 1))^n$.

   Note: X is an exponential random variable. Nevertheless, this observation should not be used in your proof.

   Solution: The memoryless property can be equivalently written as

   $$P(X > x + y) = P(X > x)P(X > y), \quad \forall x \geq 0 \ \text{ and } \ \forall y \geq 0.$$

   For $x = y = 1$, $P(X > 2) = (P(X > 1))^2$ and recursively or by induction, $P(X > n) = (P(X > 1))^n$.

2. **[20 points]** Let X, Y be two random variables with joint pdf

   $$f_{XY}(x,y) = \begin{cases} ce^{-(x+y)}, & 0 < x < \infty, 0 < y < \infty \\ 0, & \text{else} \end{cases}.$$

   (a) **[8 points]** Find c.
   Solution: $1 = \int_0^\infty \int_0^\infty ce^{-(x+y)} dxdy = c \int_0^\infty e^{-x} dx \int_0^\infty e^{-y} dy = c.$

   (b) **[12 points]** Find $b > 0$ such that $P(X > bY) = 1/3$.
   Solution: $\frac{1}{3} = \int_0^\infty \int_{by}^\infty e^{-x-y} dxdy = \int_0^\infty e^{-(b+1)y} dy = \frac{1}{b+1}$. Then we have $b = 2$.

3. **[16 points]** Let X be the number of times that a bent coin, flipped 100 times, lands heads. The bent coin lands heads with probability 0.2. Find the probability that $X \geq 50$ using the normal approximation for a binomial distribution with continuity correction. Express your solution in terms of the Q or the Φ function.

   Solution: For a binomial random variable X with $n = 100$ and $p = 0.2$, $\mathbb{E}[X] = np = 20$, and $\sigma(X) = \sqrt{np(1-p)} = 4$. Continuity correction: $P\{X \geq k\} \approx P\{\tilde{X} \geq k - 0.5\}$. Apply the continuity correction here, we have

$$\begin{aligned} P\{X \geq 50\} &\approx P\{\tilde{X} \geq 49.5\} \\ &=\approx P\left\{\frac{\tilde{X} - 20}{4} \geq \frac{49.5 - 20}{4}\right\} \\ &= Q\left(7\frac{3}{8}\right) = \Phi\left(-7\frac{3}{8}\right) \end{aligned}$$

4. **[16 points]** Find a function g so that if U is uniformly distributed over the interval $[0,1]$, $Y = g(U)$ follows the distribution with pdf

$$f_Y(c) = \begin{cases} c & 0 \le c < 1 \\ 2-c, & 1 \le c < 2 \\ 0, & \text{else.} \end{cases}$$

Solution: The CDF of the random variable Y is

$$F_Y(c) = \begin{cases} \frac{1}{2}c^2 & 0 \le c < 1 \\ 2c - \frac{1}{2}c^2 - 1, & 1 \le c < 2 \\ 0, & c < 0 \\ 1, & c \ge 2. \end{cases}$$

The function $g(U) = F_Y^{-1}(U)$. The inverse of F_Y is

$$F_Y^{-1}(u) = \begin{cases} \sqrt{2u} & 0 \le u < \frac{1}{2}, \\ 2 - \sqrt{2-2u}, & \frac{1}{2} \le u \le 1 \end{cases}$$

5. **[20 points]** Let T denote the lifetime of a light bulb. Suppose that the bulb either belongs to the factory H_0 or to the factory H_1. If the bulb belongs to the factory H_0, then T has exponential distribution with parameter $\lambda = 1$, and if it belongs to the factory H_1, then T has uniform distribution on $[0, e]$. Suppose that $\pi_0 = 1/3$ and $\pi_1 = 2/3$. Find the ML and MAP decision rules for the factory that the lamp belongs to based on T.

Solution: The likelihood ratio $\Lambda(T = t) = f_1(t)/f_0(t)$ is given as follows:

$$\Lambda(t) = \begin{cases} \frac{\frac{1}{e}}{e^{-t}} = e^{t-1}, & \text{if } t \in [0, e] \\ 0, & \text{if } t > e \end{cases}. \tag{1}$$

The ML decision rule declares H_1 if $\Lambda(t) \geq 1$, and declares H_0 otherwise; in other words, it declares H_1 if $t \in [1, e]$ and H_0 otherwise.

The MAP decision rule declares H_1 if $\Lambda(t) \geq 0.5 = \pi_0/\pi_1$, and declares H_0 otherwise; more precisely, it declares H_1 for $t \in [0.31, e]$ and H_0 otherwise.

6. **[16 points]** Suppose that customers arrive at a coffee shop according to a Poisson process with rate $\lambda = 1$. Suppose also that the coffee shop opens at time 0, and let N_t denote the number of customers that arrive in the interval $[0, t]$. Without calculating exact values, answer the following.

   (a) **[5 points]** Find $P\{N_6 = 20 | N_4 = 10\}$.
   Solution: Using the independent increment property of N_t, we have

$$P\{N_6 = 20 | N_4 = 10\} = P\{N_6 - N_4 = 10 | N_4 = 10\} = P\{N_6 - N_4 = 10\} = \frac{e^{-2} 2^{10}}{10!}$$

   (b) **[7 points]** Find $P\{N_4 = 10,\ N_8 = 30 | N_6 = 20\}$.
   Solution: Using the independent increment property of N_t, we have

$$\begin{aligned} P\{N_4 = 10,\ N_8 = 30 | N_6 = 20\} &= \frac{P\{N_4 = 10,\ N_8 = 30,\ N_6 = 20\}}{P\{N_6 = 20\}} \\ &= \frac{P\{N_4 = 10,\ N_6 - N_4 = 10,\ N_8 - N_6 = 10\}}{P\{N_6 = 20\}} \\ &= \frac{\frac{e^{-4} 4^{10}}{10!} \times \frac{e^{-2} 2^{10}}{10!} \times \frac{e^{-2} 2^{10}}{10!}}{\frac{e^{-6} 6^{20}}{20!}} \end{aligned}$$

   (c) **[4 points]** Let T_k denote the arrival time of the kth customer. What is the probability that $T_3 > 4$ (express your answer in the form of a summation.)
   Solution:

$$P\{T_3 > 4\} = P\{N_4 \le 2\} = \sum_{k=0}^{2} P\{N_4 = k\} = \sum_{k=0}^{2} \frac{e^{-4} 4^k}{k!}$$