University of Illinois Spring 2024

ECE 313: Exam II Conflict

Tuesday, April 9, 2024
7:00 p.m. — 8:15 p.m.

1. (a) Let X denote the number of applications with critical bugs out of the 3 samples. Let H_0 denote the hypothesis that the company's claim is true, and H_1 denote the hypothesis that the software testers' hypothesis is true. If H_0 is true, $X \sim \text{Bi}(3, \frac{1}{5})$; otherwise, if H_1 is true, $X \sim \text{Bi}(3, \frac{2}{5})$. The likelihood matrix is then: The ML decision rule is thus

	$X = 0$	$X = 1$	$X = 2$	$X = 3$
H_0	64/125	48/125	12/125	1/125
H_1	27/125	54/125	36/125	8/125

to declare H_0 if $X = 0$ and declare H_1 if $X = 1, 2, 3$.

(b) According to your definition of H_0 and H_1, your $p_{false\,alarm}$ may equal the value of p_{miss} in this solution, and vice versa. As long as your definitions are consistent, however, your p_e should be the same.

$$p_{false\,alarm} = P(declare\,H_1|H_0) \tag{1}$$
$$= P(X = 1, 2, 3|H_0) \tag{2}$$
$$= \frac{61}{125} \tag{3}$$

$$p_{miss} = P(declare\,H_0|H_1) \tag{4}$$
$$= P(X = 0|H_1) \tag{5}$$
$$= \frac{27}{125} \tag{6}$$

$$p_e = p_{false\,alarm} \cdot \frac{1}{3} + p_{miss} \cdot \frac{2}{3} \tag{7}$$
$$= \frac{61}{375} + \frac{18}{125} \tag{8}$$
$$= \frac{23}{75} \tag{9}$$

(c) The MAP joint probability matrix is then:

	$X=0$	$X=1$	$X=2$	$X=3$
H_0	64/375	48/375	12/375	1/375
H_1	54/375	108/375	72/375	16/375

Therefore H_0 is chosen when $X=0$ and H_1 is chosen $X=1,2,3$

(d) We notice that

$$P_{false} = \mathbf{Pr}\{H_1 \text{ claimed}|H_0\} = \mathbf{Pr}\{X \in \{1,2,3\}|H_0\} = \frac{48+12+1}{125} = \frac{61}{125}$$

and

$$P_{miss} = \mathbf{Pr}\{H_0 \text{ claimed}|H_1\} = \mathbf{Pr}\{X \in \{0\}|H_1\} = \frac{27}{125}$$

Thus

$$P_e = \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss} = \frac{115}{375} = \frac{23}{75}$$

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It is reasonable that this equals the p_e in (b) since the choices of hypotheses for different decision rules are the same.

2. (a)

$$P(N_{60}=10) = \frac{e^{-60\lambda}(60\lambda)^{10}}{10!} = \frac{180^{10}}{10!}e^{-180}$$

(b)

$$\begin{aligned} & P(N_{10}=0 \cap N_{30}-N_{10}=0 \cap N_{40}-N_{30}=0) \\ &= P(N_{10}=0)P(N_{30}-N_{10}=0)P(N_{40}-N_{30}=0) \\ &= \frac{e^{-10\lambda}(10\lambda)^0}{0!}\frac{e^{-20\lambda}(20\lambda)^0}{0!}\frac{e^{-10\lambda}(10\lambda)^0}{0!} \\ &= e^{-120} \end{aligned}$$

or noticing that $\{N_{10}=0 \cap N_{30}-N_{10}=0 \cap N_{40}-N_{30}=0\} \iff \{N_{40}=0\}$,

$$\begin{aligned} & P(N_{10}=0 \cap N_{30}-N_{10}=0 \cap N_{40}-N_{30}=0) \\ &= P(N_{40}=0) \\ &= \frac{e^{-40\lambda}(40\lambda)^0}{0!} \\ &= e^{-120} \end{aligned}$$

(c) Let A denote the event that the Geiger counter recorded one arrival in each of the three time intervals $[0,3]$, $[1,4]$, and $[2,5]$ seconds. There are three ways to meet such requirements:

	$[0,1)$	$[1,2)$	$[2,3]$	$(3,4]$	$(4,5]$
Case #1	1	0	0	1	0
Case #2	0	1	0	0	1
Case #3	0	0	1	0	0

$$\begin{aligned} P(A) &= 2 \times \left(\frac{e^{-\lambda}\lambda^1}{1!}\right)^2 \left(\frac{e^{-\lambda}\lambda^0}{0!}\right)^3 + \left(\frac{e^{-\lambda}\lambda^1}{1!}\right)^1 \left(\frac{e^{-\lambda}\lambda^0}{0!}\right)^4 \\ &= 21e^{-15} \end{aligned}$$

3. (a) We have

$$(\text{number of times a number} > 4 \text{ shows}) = 100 - X.$$

Therefore

$$A = \{X \le 162 - X + 4\} = \{X \le 83\}.$$

Since a number ≤ 4 (success) occurs with probability 2/3 on every die roll, $X \sim \text{Binom}(162, 2/3)$.

(b) Since $X \sim \text{Binom}(162, 2/3)$, therefore

$$P(A) = P\{X \le 83\} = \sum_{k=0}^{83} \binom{162}{k} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{162-k} = \left(\frac{1}{3}\right)^{162} \sum_{k=0}^{83} \binom{162}{k} 2^k.$$

(c) Note that $E[X] = 108$, $\text{Var}(X) = (162)p(1-p) = 36$, and $\sigma = \sqrt{\text{Var}(X)} = 6$. Therefore

$$\begin{aligned}
P(A) = P\{X \le 83\} &\approx P\{\tilde{X} \le 83.5\} \\
&= P\left\{\frac{\tilde{X} - 108}{6} \le \frac{83.5 - 108}{6}\right\} \\
&= \Phi\left(\frac{83.5 - 108}{6}\right) = \Phi(-\frac{24.5}{6}) = \Phi(-4.08) = Q(4.08).
\end{aligned}$$

where the approximation comes from Gaussian approximation with continuity correction.

(d) We have

$$\begin{aligned}
P(B) &= P\{X = 55\} = P\{54.5 < X < 55.5\} \\
&\approx P\{54.5 < \tilde{X} < 55.5\} \\
&= P\left\{\frac{54.5 - 108}{6} < \frac{\tilde{X} - 108}{6} < \frac{55.5 - 108}{6}\right\} \\
&= Q\left(\frac{54.5 - 108}{6}\right) - Q\left(\frac{55.5 - 108}{6}\right) = Q(-8.91) - Q(-8.75) \\
&= \Phi(8.91) - \Phi(8.75).
\end{aligned}$$

4. (a) $f_X(x)$ must be non-negative for all real x. Because c is positive, $x(1-x)$ must be nonnegative for $x \in (a, b)$.

$$x(1-x) \geq 0 \tag{10}$$
$$x \geq x^2 \tag{11}$$

This is true when $x \in [0, 1]$. Therefore, for the *pdf* to be nonnegative,

$$0 \leq a < b \leq 1$$

(b)

$$\int_a^b cx(1-x)dx = \int_a^b cx - cx^2 dx \tag{12}$$
$$= c(\frac{x^2}{2} - \frac{x^3}{3})|_a^b \tag{13}$$
$$= c(b^2(\frac{1}{2} - \frac{b}{3}) - a^2(\frac{1}{2} - \frac{a}{3})) \tag{14}$$

We know this PDF must integrate to 1 over its support, so setting this equal to 1 gives:

$$c = (b^2(\frac{1}{2} - \frac{b}{3}) - a^2(\frac{1}{2} - \frac{a}{3}))^{-1}$$

(c)

$$\int_{\frac{1}{2}}^{\infty} f_X(x)dx = \int_{\frac{1}{2}}^b cx - cx^2 dx \tag{15}$$
$$= c(\frac{x^2}{2} - \frac{x^3}{3})|_{\frac{1}{2}}^b \tag{16}$$
$$= c(b^2(\frac{1}{2} - \frac{b}{3}) - \frac{1}{4}(\frac{1}{2} - \frac{1}{6})) \tag{17}$$
$$= c(b^2(\frac{1}{2} - \frac{b}{3}) - \frac{1}{12}) \tag{18}$$
$$\tag{19}$$

Substituting the value of c found before gives

$$P(X > \frac{1}{2}) = \frac{b^2(\frac{1}{2} - \frac{b}{3}) - \frac{1}{12}}{b^2(\frac{1}{2} - \frac{b}{3}) - a^2(\frac{1}{2} - \frac{a}{3})}$$