1. (8 points) Find a formula for $f(x)$ given that $f''(x) = 5\sin x + 3\cos x$, $f(0) = 10$, and $f'(0) = 10$.

$f''(x) = 5\sin x + 3\cos x$ has antiderivative

$f'(x) = -5\cos x + 3\sin x + C$

$f'(0) = 10 \Rightarrow C = 15$

$f'(x) = -5\cos x + 3\sin x + 15$

has antiderivative

$f(x) = -5\sin x - 3\cos x + 15x + D$

$f(0) = 10 \Rightarrow D = 13$

$f(x) = -5\sin x - 3\cos x + 15x + 13$

2. (6 points) The population of a town is currently 400, but it is expected to increase at a rate of $200e^{0.5t}$ people per year where t represents the number of years from now. What is the population of this town expected to be in 10 years?

Population in 10 years = current Population + change in population

$$= 400 + \int_0^{10} 200e^{0.5t}\,dt$$

$$= 400 + \int_0^{5} 400e^{u}\,du \quad \left(\begin{array}{l} u = 0.5t \\ du = 0.5dt \end{array}\right)$$

$$= 400 + \left[400e^{u}\right]_0^5$$

$$= 400 + (400e^5 - 400e^0)$$

$$= 400e^5 \text{ people}$$

3. (6 points) Evaluate the following limit.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5k}{n^3}+\frac{7}{n}\right) = \lim_{n\to\infty}\left(\frac{5}{n^3}\sum_{k=1}^{n}k + \frac{7}{n}\sum_{k=1}^{n}1\right)$$

$$= \lim_{n\to\infty}\left(\frac{5}{n^3}\cdot\frac{n(n+1)}{2} + \frac{7}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{5n+5}{2n^2}+7\right) = 0+7 = 7$$

4. (6 points) The definite integral $\int_2^6 e^{t^2}\,dt$ can be written as a limit. Fill in the missing information in this limit.

$$\int_2^6 e^{t^2}\,dt = \lim_{n\to\infty}\sum_{k=1}^{n}\left[e^{(2+\frac{4k}{n})^2}\cdot\frac{4}{n}\right]$$

$$\Delta x = \frac{6-2}{n} = \frac{4}{n}$$

$$x_k = 2 + k\cdot\Delta x = 2 + k\cdot\frac{4}{n}$$

$$f(x_k) = e^{(2+\frac{4k}{n})^2}$$

5. (12 points) Suppose that f is an odd function and g is an even function which are each integrable on the interval $[-5,5]$. Given that $\int_0^5 f(x)\,dx = 8$ and $\int_0^5 g(x)\,dx = 3$, evaluate the following definite integrals.

(a) $\int_5^0 g(x)\,dx = -\int_0^5 g(x) = -3$

(b) $\int_5^5 f(x)\,dx = 0$

(c) $\int_{-5}^5 (2f(x)+4g(x))\,dx = 2\int_{-5}^5 f(x)dx + 4\int_{-5}^5 g(x)$

$= 2\cdot 0 + 4\cdot(2\cdot 3) = 24$

(d) $\int_{-5}^5 \left(4+(f(x))^3\right)\,dx = \int_{-5}^5 4dx + \int_{-5}^5 (f(x))^3 dx$

$= 40 + 0 = 40$ since f^3 is odd

6. (6 points each) Evaluate the following ~~definite and indefinite integrals~~.

(a) $\int \left(e^x + \frac{1}{3x} + 5\right)\, dx$ $= e^x + \frac{1}{3}\ln|x| + 5x + C$

note: $\frac{1}{3}\ln|3x|$ is not the antiderivative of $\frac{1}{3x}$
however $\frac{1}{3}\ln|3x| = \frac{1}{3}(\ln 3 + \ln|x|) = \frac{1}{3}\ln 3 + \frac{1}{3}\ln|x|$
since $\frac{1}{3}\ln 3$ is a constant, this incorrect approach still yields the same family of functions in my answer

(b) $\int_1^2 (10x+5)\, dx$ $= \left(5x^2 + 5x\right)\Big|_1^2$

$= (5\cdot 4 + 5\cdot 2) - (5+5)$

$= 30 - 10$

$= 20$

(c) $\int_0^2 \left(3 + 2e^{-x}\right)\, dx$ $= \left(3x - 2e^{-x}\right)\Big|_0^2$

$= (6 - 2e^{-2}) - (0 - 2e^{0})$

$= (6 - 2e^{-2}) + 2$

$= 8 - 2e^{-2}$

(d) $\int x^2\sqrt{x^3+4}\, dx$ $= \int \frac{1}{3} u^{1/2}\, du = \frac{1}{3}\cdot\frac{2}{3} u^{3/2} + C$

$u = x^3 + 4$
$du = 3x^2\, dx$

$= \frac{2}{9}(x^3+4)^{3/2} + C$

7. (5 points each) Evaluate the following indefinite integrals.

(a) $\int x^2(x+4)^{10}\,dx \quad = \int (u-4)^2 u^{10}\,du$

$u = x+4$ so $x = u-4$

$du = dx$

$$= \int (u^2 - 8u + 16)u^{10}\,du$$

$$= \int (u^{12} - 8u^{11} + 16u^{10})\,du$$

$$= \frac{1}{13}u^{13} - \frac{8}{12}u^{12} + \frac{16}{11}u^{11} + C$$

$$= \frac{1}{13}(x+4)^{13} - \frac{2}{3}(x+4)^{12} + \frac{16}{11}(x+4)^{11} + C$$

(b) $\int \sec^6 x \tan^3 x\,dx$ (MANY APPROACHES)

METHOD 1)

$$\int \sec^6 x \tan^3 x\,dx = \int \sec^5 x \tan^2 x \sec x \tan x\,dx$$

$$= \int \sec^5 x(\sec^2 x - 1)\sec x \tan x\,dx$$

$$= \int u^5(u^2-1)\,du$$ (where $u = \sec x$, $du = \sec x \tan x\,dx$)

$$= \int (u^7 - u^5)\,du$$

$$= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{8}\sec^8 x - \frac{1}{6}\sec^6 x + C$$

METHOD 2

$$\int \sec^6 x \tan^3 x\,dx = \int \sec^4 x \tan^3 x \sec^2 x\,dx$$

$$= \int (\sec^2 x)^2 \tan^3 x \sec^2 x\,dx$$

$$= \int (\tan^2 x + 1)^2 \tan^3 x \sec^2 x\,dx$$

$$= \int (u^2+1)^2 u^3\,du \quad \left(\begin{array}{l}\text{where} \\ u = \tan x \\ du = \sec^2 x\,dx\end{array}\right)$$

$$= \int (u^4 + 2u^2 + 1) u^3\,du$$

$$= \int (u^7 + 2u^5 + u^3)\,du$$

$$= \frac{1}{8}u^8 + \frac{2}{6}u^6 + \frac{1}{4}u^4 + C$$

$$= \frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

METHOD 3

$$\int \sec^6 x \tan^3 x\,dx = \int \frac{1}{\cos^6 x} \cdot \frac{\sin^3 x}{\cos^3 x}\,dx$$

$$= \int \frac{\sin^3 x}{\cos^9 x}\,dx$$

$$= \int \frac{\sin^2 x}{\cos^9 x} \cdot \sin x\,dx$$

$$= \int \frac{1 - \cos^2 x}{\cos^9 x} \sin x\,dx$$

$$= \int \frac{1-u^2}{u^9}(-du) \quad \left(\begin{array}{l}\text{where} \\ u = \cos x \\ du = -\sin x\,dx\end{array}\right)$$

$$= \int (-u^{-9} + u^{-7})\,du$$

$$= \frac{1}{8}u^{-8} - \frac{1}{6}u^{-6} + C = \frac{1}{8\cos^8 x} - \frac{1}{6\cos^6 x} + C$$

METHOD 4

$$\int \sec^6 x \tan^3 x \, dx = \int \sec^4 x \tan^2 x \sec^2 x \tan x \, dx$$

$$= \int \sec^4 x (\sec^2 x - 1) \sec^2 x \tan x \, dx$$

$$= \int u^2 (u-1) \tfrac{1}{2} du$$

where
$u = \sec^2 x$
$du = 2 \sec x \sec x \tan x \, dx$
or
$\tfrac{1}{2} du = \sec^2 x \tan x \, dx$

$$= \int \left(\tfrac{1}{2} u^3 - \tfrac{1}{2} u^2\right) du$$

$$= \tfrac{1}{8} u^4 - \tfrac{1}{6} u^3 + C$$

$$= \tfrac{1}{8} \sec^8 x - \tfrac{1}{6} \sec^6 x + C$$

METHOD 5

$$\int \sec^6 x \tan^3 x \, dx = \int (\sec^2 x)^2 \tan^2 x \tan x \sec^2 x \, dx$$

$$= \int (\tan^2 x + 1)^2 \tan^2 x \tan x \sec^2 x \, dx$$

$$= \int (u+1)^2 u \left(\tfrac{1}{2} du\right)$$

where
$u = \tan^2 x$
$du = 2 \tan x \sec^2 x \, dx$
or
$\tfrac{1}{2} du = \tan x \sec^2 x \, dx$

$$= \int (u^2 + 2u + 1) \cdot \tfrac{1}{2} u \, du$$

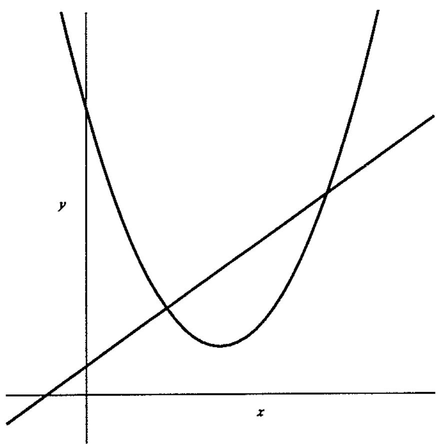
$$= \int \left(\tfrac{1}{2} u^3 + u^2 + \tfrac{1}{2} u\right) du$$

$$= \tfrac{1}{8} u^4 + \tfrac{1}{3} u^3 + \tfrac{1}{4} u^2 + C$$

$$= \tfrac{1}{8} \tan^8 x + \tfrac{1}{3} \tan^6 x + \tfrac{1}{4} \tan^4 x + C$$

8. (6 points each) Let **R** be the region bounded by the graphs of $f(x) = x^2 - 10x + 30$ and $g(x) = 2x + 3$ as shown below. Set up, but do not evaluate, definite integrals which represent the given quantities. Use proper notation.

$$x^2 - 10x + 30 = 2x + 3$$
$$x^2 - 12x + 27 = 0$$
$$(x-3)(x-9) = 0$$
$$x - 3 = 0 \text{ or } x - 9 = 0$$
$$x = 3 \text{ or } x = 9$$



(a) The area of **R**.

$$A = \int_3^9 ((2x+3) - (x^2 - 10x + 30))\,dx$$
$$= \int_3^9 (-x^2 + 12x - 27)\,dx$$

(b) The volume of the solid obtained when **R** is revolved around the y-axis.

$$V = \int_3^9 2\pi x\,((2x+3) - (x^2 - 10x + 30))\,dx$$

(c) The volume of the solid obtained when **R** is revolved around the horizontal line $y = -10$.

$$\cancel{V = \int_3^9 \left(\pi(2x+3)^2 - \pi(x^2 - 10x + 30)^2\right)dx}$$

oops - this is around x-axis

$$V = \int_3^9 \left(\pi(2x+3+10)^2 - \pi(x^2 - 10x + 30 + 10)^2\right)dx$$

9. (5 points each) Set up, but do not evaluate, definite integrals which represent the given quantities. Use proper notation.

(a) The average value of $f(x) = \dfrac{e^{3x}}{\pi - 2}$ ~~on the interval~~ $[3, \pi]$.

$$f_{ave} = \frac{1}{\pi - 3} \int_3^{\pi} \frac{e^{3x}}{\pi - 2}\, dx$$

(b) The length of the curve $f(x) = 3\sin 2x$ for $0 \le x \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{1 + (f'(x))^2}\, dx$$

$$= \int_0^{2\pi} \sqrt{1 + (6\cos(2x))^2}\, dx$$

$$= \int_0^{2\pi} \sqrt{1 + 36\cos^2(2x)}\, dx$$

we used $f'(x) = 3\cos(2x) \cdot 2$

$= 6\cos(2x)$