Math 231 Exam 1

UIUC, October 25, 2010

1 Compute $\int xe^{5x}\,dx$.

2 Compute $\int \sin^2(3t)\,dt$.

3 Compute $\int \frac{1}{\sqrt{x^2-4}}\,dx$

4 Compute $\int \tan^3\theta \sec\theta\,d\theta$.

5 Compute $\int_1^\infty \frac{1}{(3x+1)^2}\,dx$

6 Compute $\int \frac{1}{x^2-2x+5}\,dx$

7 Compute $\int \frac{1}{x^2-6x+5}\,dx$

8 Write out the form of the partial fraction decomposition of the function

$$\frac{x^4+1}{x^5+4x^3}$$

Do not determine the numerical values of the coefficients.

9 Use the Comparison Theorem to determine if the following converges or diverges.

$$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$$

10 For any continuous function f on $[a,b]$, $P(f)$ will approximate $\int_a^b f(x)\,dx$ to an error no more than $\frac{K_0(b-a)^3}{5}$ when $|f(x)| \le K_0$ for all x in $[a,b]$. You use $P(f)$ to numerically approximate the integral $\int_0^1 \sin x\,dx$ by subdividing the interval into 100 equal pieces and applying $P(f)$ to each of the smaller intervals. What is an upper bound for the error of your approximation and why?

11 Compute $\int \frac{1}{1+e^x}\,dx$

12 Compute $\int \tan^{-1}(\sqrt{x})\,dx$

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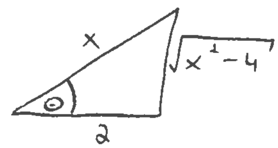
1) $\int x \cdot e^{5x}\,dx = \frac{x \cdot e^{5x}}{5} - \int \frac{e^{5x}}{5}\,dx$ with $u = x$, $du = dx$, $dv = e^{5x}dv$, $v = \frac{e^{5x}}{5}$

$= \frac{x \cdot e^{5x}}{5} - \frac{e^{5x}}{25} + C$

2) $\int \sin^2(3t)\,dt = \int\left(\frac{1}{2} - \frac{1}{2}\cos(6t)\right)dt$ Recall: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$= \frac{1}{2}t - \frac{1}{12}\sin(6t) + C$

3) $\int \frac{1}{\sqrt{x^2-4}}\,dx$ $x = 2\sec\Theta$, $dx = 2\tan\Theta\sec\Theta\,d\Theta$



$\tan\Theta = \frac{\sqrt{x^2-4}}{2}$

$\int \frac{2\tan\Theta\sec\Theta\,d\Theta}{2\tan\Theta} = \int \sec\Theta\,d\Theta = \ln|\sec\Theta + \tan\Theta| + C$

$= \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$

4) $\int \tan^3\Theta\sec\Theta\,d\Theta = \int \tan^2\Theta \cdot \tan\Theta \cdot \sec\Theta\,d\Theta = \int(\sec^2\Theta - 1)\tan\Theta \cdot \sec\Theta\,d\Theta$ $u = \sec\Theta$, $du = \tan\Theta\sec\Theta\,d\Theta$

$= \int(u^2-1)\,du = \frac{u^3}{3} - u + C = \frac{\sec^3\Theta}{3} - \sec\Theta + C$

5) $\int_1^\infty \frac{1}{(3x+1)^2}\,dx$ $u = 3x+1$, $du = 3dx$

$\frac{1}{3}\int_4^\infty \frac{du}{u^2} = \frac{1}{3}\left[-\frac{1}{x}\right]_4^\infty = \frac{1}{3}\left(0 + \frac{1}{4}\right) = \frac{1}{12}$

6) $\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{x^2-2x+1+4}\,dx = \int \frac{1}{(x-1)^2+4}\,dx = \frac{1}{4}\int \frac{dx}{\left(\frac{x-1}{2}\right)^2+1}$ $u = \frac{x-1}{2}$, $du = \frac{1}{2}dx$

$= \frac{1}{2}\int \frac{du}{u^2+1} = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan\left(\frac{x-1}{2}\right) + C$

7) $\int \frac{1}{x^2-6x+5}\,dx = \int \frac{1}{(x-5)(x-1)}\,dx = \int\left(\frac{A}{x-5} + \frac{B}{x-1}\right)dx$

$Ax - A + Bx - 5B = 1$

$A + B = 0$
$-A - 5B = 1$ +

$A + B = 0$
$-4B = 1 \rightarrow \boxed{B = -\frac{1}{4}}$
$\boxed{A = +\frac{1}{4}}$

$= \int\left(\frac{1}{4} \cdot \frac{1}{x-5} - \frac{1}{4} \cdot \frac{1}{x-1}\right)dx$

$= \frac{1}{4}\ln|x-5| - \frac{1}{4} \cdot \ln|x-1| + C$

8) $\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3 \cdot (x^2+4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$

9) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx = \int_1^2 \frac{x+1}{\sqrt{x^4-x}}\,dx + \underbrace{\int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx}$ because the integral is improper both at 1 and ∞

$\frac{x+1}{\sqrt{x^4-x}} > \frac{x}{\sqrt{x^4-x}} > \frac{x}{\sqrt{x^4}} = \frac{x}{x^2} = \frac{1}{x}$

The original function is greater than $\frac{1}{x}$ which is already divergent on $[2,\infty)$, so the overall integral is divergent.

10) $\int_0^1 \sin x \, dx$ approximated by subdividing into 100 equal pieces $\rightarrow \Delta x = \frac{1}{100}$

The error for each subinterval is bounded by $\frac{K_0 \cdot \left(\frac{1}{100}\right)^3}{5}$

Summing over all 100 subintervals : $\frac{K_0 \cdot \left(\frac{1}{100}\right)^3}{5} \cdot 100 = \underline{\underline{\frac{K_0}{50,000}}}$

11) $\int \frac{1}{1+e^x} \, dx$

$u = 1+e^x$

$du = e^x dx \rightarrow dx = \frac{du}{e^x} = \frac{du}{u-1}$

$= \int \frac{1}{u} \cdot \frac{du}{u-1} = \int \frac{1}{u(u-1)} \, du = \int \left(\frac{A}{u} + \frac{B}{u-1}\right) du$

$Au - A + Bu = 1$

$A + B = 0$

$-A = 1 \rightarrow \boxed{A = -1}$

$\boxed{B = 1}$

$= \int \left(-\frac{1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C$

$= -\ln|1+e^x| + \ln|1+e^x-1| + C = \ln|e^x| - \ln|1+e^x| + C$

$= x - \ln(e^x+1) + C$

12) $\int \arctan\sqrt{x} \, dx$

$z = \sqrt{x}$

$dz = \frac{1}{2\sqrt{x}} dx \rightarrow dx = 2z \, dz$

$= 2 \cdot \int z \cdot \arctan z \, dz$

$u = \arctan z \qquad dv = z$

$du = \frac{1}{1+z^2} dz \qquad v = \frac{z^2}{2}$

$= 2 \cdot \left(\frac{z^2}{2} \cdot \arctan z - \int \frac{z^2}{2} \cdot \frac{1}{1+z^2} dz\right)$

$= z^2 \arctan z - \int \frac{z^2}{1+z^2} dz = z^2 \cdot \arctan z - \int \frac{z^2+1-1}{z^2+1} dz = z^2 \cdot \arctan z - \int \left(1 - \frac{1}{z^2+1}\right) dz$

$= z^2 \cdot \arctan z - z + \int \frac{1}{z^2+1} dz = z^2 \cdot \arctan z - z + \arctan z + C$

$= x \cdot \arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C$