Sample Exam 2 Solutions

1. (15 points) The following are worth 3 points each and are short answer.

   (a) The series $\sum_{n=1}^{\infty} a_n$ has the property that the n-th partial sum for every n is given by

   $$s_n = \frac{n(n+1)}{2} = 1 + 2 + 3 + \cdots + n$$

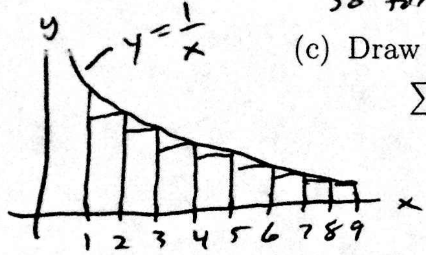
   What is a_n for every n? $a_n = s_n - s_{n-1} = (1 + \cdots + n) - (1 + \cdots + n-1) = \boxed{n}$

   (b) Show that for any number $r \neq 1$,

   $$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

   $$(1-r)(1 + r + r^2 + \cdots + r^n) = (1 + r + r^2 + \cdots + r^n) - (r + r^2 + \cdots + r^{n+1})$$
   $$= 1 - r^{n+1}$$

   so for all $r \neq 1$, dividing through by $1-r$ yields $1 + r + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$



   (c) Draw a diagram and give a brief explanation why $\sum_{n=2}^{9} \frac{1}{n}$ is less than $\int_1^9 \frac{1}{x}\,dx$.

   $\sum_{n=2}^{9} \frac{1}{n}$ is the cumulative area of the rectangles, which is less than the area under $y = \frac{1}{x}$ from 1 to 9.

   (d) The series $\sum_{n=1}^{\infty} a_n$ has the property that its n-th partial sum $s_n = a_1 + a_2 + \cdots + a_n$ for every n is $\frac{1}{n}$. Does the series converge and if so to what value?

   The series converges to $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{n} = \boxed{0}$

   (e) Give~~n~~ an example of a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ that diverges while the series $\sum_{n=1}^{\infty}(a_{2n-1} + a_{2n}) = (a_1 + a_2) + (a_3 + a_4) + (a_5 + a_6) + \cdots$ converges.

   Let $a_n = (-1)^n$. Then $\sum_{n=1}^{\infty} a_n$ diverges because the sequence of partial sums is $(-1, 0, -1, 0, -1, 0, \ldots)$, which does not converge. However,

   $$\sum_{n=1}^{\infty} (a_{2n-1} + a_{2n}) = \sum_{n=1}^{\infty} ((-1)^{2n-1} + (-1)^{2n}) = \sum_{n=1}^{\infty} (-1+1) = \sum_{n=1}^{\infty} 0 = 0.$$

2. (10 points) Show that the following series diverges except for one value of c. Then compute the sum of the series for that value of c.

$$\sum_{n=1}^{\infty}\left(\frac{c}{2n}+\frac{1}{n+1}\right)$$

Let $a_n = \frac{c}{2n} + \frac{1}{n+1} = \frac{(2+c)n + c}{2n(n+1)}$ and let $A_n = -a_n$. Observe $\sum A_n$ converges $\Leftrightarrow \sum a_n$ converges, so we can work with either series.

I. If $c > -2$, $a_n \geq 0$ for all $n \geq \frac{-c}{2+c}$ and

$\lim\limits_{n\to\infty} \frac{a_n}{n^{-1}} = \lim\limits_{n\to\infty} \frac{(2+c)n+c}{2(n+1)} = \frac{2+c}{2} > 0$ so by the LCT,

since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} a_n$.

II. If $c < -2$, $a_n < 0$ for all n so $A_n = -a_n > 0$ for all n, and $\lim\limits_{n\to\infty} \frac{A_n}{n^{-1}} = \lim\limits_{n\to\infty} \frac{-[(2+c)n+c]}{2(n+1)} = \frac{-(2+c)}{2} > 0$ so by the LCT, since $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum\limits_{n=1}^{\infty} A_n$ (and so does $\sum\limits_{n=1}^{\infty} a_n$).

III. If $c = -2$, $a_n = \frac{-2}{2n} + \frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{n}$ so

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty}\left(\cancel{\tfrac{1}{2}} - 1 + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{2}} + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{3}} + \cdots + \cancel{\tfrac{1}{N}} - \cancel{\tfrac{1}{N-1}} + \tfrac{1}{N+1} - \cancel{\tfrac{1}{N}}\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{N+1} - 1\right) = \boxed{-1}$$

3. (10 points, 5 points each) Find the sum of the following series, if they exist, otherwise explain:

(a) $5 + 2 + \frac{20}{25} + \frac{40}{125} + \dots$ $= 5\left(1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots\right)$

$$= 5 \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = 5 \cdot \frac{1}{1 - 2/5}$$

$$= \boxed{\frac{25}{3}}$$

(b) $\sum_{n=0}^{\infty} \frac{11^n}{(-8)^{n-1}}$. $= \sum_{n=0}^{\infty} (-8) \frac{11^n}{(-8)^n} = (-8) \sum_{n=0}^{\infty} \left(\frac{11}{-8}\right)^n$

Since $|r| = \left|\frac{11}{-8}\right| > 1$, the geometric series diverges.

4. (10 points, 5 points each)

(a) Does the *sequence* $\{\frac{3n^2+2}{4n^2-4}\}$ converge? If so, find the limit, if not, explain.

Yes, the sequence converges to

$$\lim_{n\to\infty} \frac{3n^2+2}{4n^2-4} = \lim_{n\to\infty} \frac{3+2/n^2}{4-4/n^2} = \boxed{\frac{3}{4}}$$

(b) Does the series $\sum_{n=4}^{\infty}(-1)^n \frac{3n^2+2}{4n^2-4}$ converge? If so, find the sum, if not, explain.

Since the sequence $a_n = \frac{3n^2+2}{4n^2-4}$ converges to $3/4$, the sequence $(-1)^n a_n$ cannot converge to 0 (for if $\lim_{n\to\infty}(-1)^n a_n = 0$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} |(-1)^n a_n| = 0$, contradicting (a)).

By the test for divergence, the given series diverges.

5. (10 points) Determine whether the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to indicate your reasoning to receive full credit.

$$\sum_{n=0}^{\infty} \frac{n}{3n^4+4}$$

Since all terms are nonnegative and

$$\lim_{n\to\infty} \frac{n}{3n^4+4} \cdot \frac{n^3}{1} = \lim_{n\to\infty} \frac{n^4}{3n^4+4} = \frac{1}{3} > 0,$$

by the LCT, since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series, $p=3$), the given series also converges. Since all terms are nonnegative, the series $\boxed{\text{converges absolutely}}$

6. (10 points) Determine whether the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to indicate your reasoning to receive full credit.

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{n^2 2^n}{n!}$$

Since

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(n+1)^2 2^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n n^2 2^n} \right| = \lim_{n\to\infty} \frac{(n+1)^2\, 2}{n^2 (n+1)} = 0 < 1,$$

the given series $\boxed{\text{converges absolutely}}$ by the ratio test.

7. (10 points) Determine whether the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to indicate your reasoning to receive full credit.

$$\sum_{n=3}^{\infty}(-1)^n\frac{1}{n\sqrt{\ln(n)}}$$

- Since the sequence $b_n = \frac{1}{n\sqrt{\ln n}}$ is a positive sequence which decreases to 0, by the AST, the given series converges.
- However, since the function $f(x) = \frac{1}{x\sqrt{\ln x}}$ is nonnegative, continuous, and decreasing to 0 on $[3,\infty)$ and

$$\int_3^{\infty}\frac{1}{x\sqrt{\ln x}}\,dx \overset{u=\ln x}{=} \int_{\ln 3}^{\infty} u^{-1/2}\,du = \lim_{t\to\infty}\left(2\sqrt{t}-2\sqrt{\ln 3}\right) = \infty$$

by the integral test,

$$\sum_{n=3}^{\infty}\frac{1}{n\sqrt{\ln n}} \text{ diverges}$$

so the given series does not converge absolutely.

- Therefore, the given series $\boxed{\text{converges conditionally}}$

8. (15 points) Determine whether *one* of the following series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to indicate your reasoning to receive full credit and which attempt you want graded if you do work on both problems.

$$\text{(a)} \quad \sum_{n=0}^{\infty} \frac{3^n + (-2)^n}{6^n \sqrt[3]{5}} \qquad \text{or} \qquad \sum_{n=1}^{\infty} \frac{n! \tan^{-1}(n)}{(2n)!} \quad \text{(b)}$$

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n+(-2)^n}{6^n \sqrt[3]{5}} = 5^{-1/3} \sum_{n=0}^{\infty} \left(\left(\tfrac{1}{2}\right)^n + \left(-\tfrac{1}{3}\right)^n\right) = 5^{-1/3}\left(\sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(-\tfrac{1}{3}\right)^n\right)$

$\displaystyle = 5^{-1/3}\left(\frac{1}{1-1/2} + \frac{1}{1+1/3}\right)$

$\displaystyle = 5^{-1/3}\left(2 + 3/4\right) = \frac{11}{4\sqrt[3]{5}}$

Since $2^n \leq 3^n$ for all n, each term of this series is nonnegative, so the series is $\boxed{\text{absolutely convergent}}$

(b) Since

$\displaystyle L = \lim_{n\to\infty} \left| \frac{(n+1)!\,\arctan(n+1)}{(2n+2)!} \cdot \frac{(2n)!}{n!\,\arctan(n)} \right| = \lim_{n\to\infty} \frac{(n+1)}{(2n+2)(2n+1)} \cdot \frac{\arctan(n+1)}{\arctan(n)}$

$\displaystyle = 0 \cdot \frac{\pi/2}{\pi/2} = 0 < 1$

the given series $\boxed{\text{converges absolutely}}$ by the ratio test.