Math 231 Exam 3

UIUC, November 17, 2010

1 Short answer.

(a) The power series $\sum_{n=0}^{\infty} a_n x^n$ is known to diverge for $x = -2$ and $x = 5$. What is the maximum radius of convergence for this series?

(b) One can show that a power series representation for $x\sin(x^2)$ is given by

$$x\sin(x^2) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + \cdots$$

What is $(x\sin(x^2))^{(11)}(0)$ and why?

2 Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \sqrt{n}x^n$$

3 Find the radius of convergence and interval of convergence of the series.

$$\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{n}$$

4 Find a series solution to the integral

$$\int_0^1 x^4 e^x \, dx$$

5

(a) How would the power series for $(1+4x)^{-3}$ be expressed using the binomial coefficient theorem?

(b) Find a power series for $\frac{1}{(1+4x)^3}$ using the fact that it is the second derivative of $\frac{1}{32(1+4x)}$.

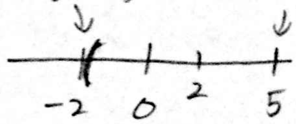
(c) Using your results from (a) and (b), what is the value of $\binom{-3}{6}$?

6 Suppose you know that for some mysterious function f, $f(3) = 1, f'(3) = 2, f^{(2)}(3) = 3, f^{(3)}(3) = 4$ while $f^{(4)}(x) = \ln x$.

a) What is the the Taylor polynomial of degree 3 centered at 3 for f?

b) If you use the the degree 3 Taylor polynomial centered at 3 to approximate $f(e)$, use Taylor's Theorem to estimate your accuracy. Be sure to justify your answer.

1. a)



The biggest R can be is $\boxed{2}$.

b) By the uniqueness of power series representations, the coefficient on x^{11} in the power series of $f(x) = x\sin(x^2)$ is $\frac{f^{(11)}(0)}{11!}$. We are given that the coefficient on x^{11} is $\frac{1}{5!}$.

Hence

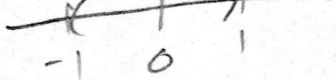
$$\frac{f^{(11)}(0)}{11!} = \frac{1}{5!} \Rightarrow f^{(11)}(0) = \boxed{\frac{11!}{5!}}$$

2. $$\lim_{n\to\infty} \left| \frac{\sqrt{n+1}\, x^{n+1}}{\sqrt{n}\, x^n} \right| = |x| \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{n}} = |x| \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = |x|$$

The series converges absolutely if $|x|<1$ by the ratio test $\Rightarrow \boxed{R=1}$.

• $\underline{x=-1}$: $\sum_{n=1}^{\infty} \sqrt{n}\,(-1)^n$

Since $\lim_{n\to\infty} \sqrt{n}\,(-1)^n$ DNE (the sequence goes to ∞ along even terms and $-\infty$ along odd terms), the series diverges by the n^{th} term test



Interval of convergence:

$\boxed{(-1,1)}$

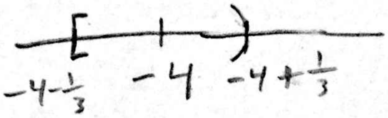
• $\underline{x=1}$: $\sum_{n=1}^{\infty} \sqrt{n}$

Since $\lim_{n\to\infty} \sqrt{n} = \infty$, the series diverges by the n^{th} term test

3) $$\lim_{n\to\infty} \left| \frac{3^{n+1}(x+4)^{n+1}}{n+1} \cdot \frac{n}{3^n(x+4)^n} \right|$$

$$= |x+4| \lim_{n\to\infty} \frac{3n}{n+1} = 3|x+4|$$

By the ratio test, the series converges absolutely if $3|x+4| < 1 \Rightarrow |x+4| < \frac{1}{3} \Rightarrow \boxed{R = 1/3}$



• $\underline{x = -4 - \frac{1}{3}}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(-1/3)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Since $\frac{1}{n}$ is decreasing and goes to 0, the series converges by the A.S.T.

• $\underline{x = -4 + \frac{1}{3}}$: $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(1/3)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$

This series diverges by the p-series test ($p=1$).

Interval of convergence:

$$\left[-4-\frac{1}{3}, -4+\frac{1}{3}\right) = \left[\frac{-13}{3}, \frac{-11}{3}\right)$$

4) $$\int_0^1 x^4 e^x\,dx = \int_0^1 x^4 \sum_{n=0}^{\infty} \frac{x^n}{n!}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^{n+4}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+5}}{n!\,(n+5)} \Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\,(n+5)}$$

5) a) $(1+4x)^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} (4x)^n$

$= \sum_{n=0}^{\infty} \binom{-3}{n} 4^n x^n \qquad |4x|<1 \Rightarrow |x|<\frac{1}{4}$

b) $\frac{1}{32(1+4x)} = \frac{1}{32} \frac{1}{1-(-4x)} = \frac{1}{32} \sum_{n=0}^{\infty} (-4x)^n \quad \left(|-4x|<1 \Rightarrow |x|<\frac{1}{4} \right)$

$= \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^n}{32}$

$\frac{d}{dx}\left(\frac{1}{32(1+4x)} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n n}{32} x^{n-1}$

$\frac{1}{(1+4x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{32(1+4x)} \right) = \sum_{n=2}^{\infty} \frac{(-1)^n 4^n n(n-1) x^{n-2}}{32}$

$\overset{k=n-2}{=} \sum_{k=0}^{\infty} \frac{(-1)^{k+2} 4^{k+2} (k+2)(k+1) x^k}{32}$

$= \sum_{k=0}^{\infty} \frac{(-1)^k 4^k (k+2)(k+1) x^k}{2}$

c) The coefficient on x^6 in (a) is $\binom{-3}{6} 4^6$ and in (b) is $\frac{(-1)^6 4^6 \cdot 8 \cdot 7}{2}$ Equating the two gives us

$\binom{-3}{6} 4^6 = \frac{(-1)^6 4^6 \cdot 8 \cdot 7}{2} \Rightarrow$

$\boxed{\binom{-3}{6} = 28}$

6) a) $T_3(x) = 1 + 2(x-3) + \frac{3}{2}(x-3)^2$
$+ \frac{4}{3!}(x-3)^3$

b) $T_3(e) = 1 + 2(e-3) + \frac{3}{2}(e-3)^2 + \frac{2}{3}(e-3)^3$

On $[e, 3]$, the maximum of $f^{(4)}(x) = \ln x$ is $\ln(3)$ (at $x=3$), since $f^{(4)}$ is increasing). By Taylor's Theorem,

$$|R_3(e)| \leq \frac{\ln(3)}{4!}|e-3|^4$$
$$= \frac{\ln(3)}{24}(3-e)^4$$

(There are other possible choices for the interval instead of $[e, 3]$).