1. Show that for any number $r \neq 1$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

2. Which of the following expressions would be used to determine the partial fraction decomposition of

$$\frac{1}{(x^2 - 1)(x^2 + 3)} \ ?$$

(a) $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+3}$ $\qquad$ (b) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+3}$

(c) $\dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C}{x+3} + \dfrac{D}{(x+3)^2}$ $\qquad$ (d) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+3} + \dfrac{D}{(x+3)^2}$

3. True or False. For every $x > 0$,

$$\sin x > x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

4. Suppose $0 \leq a_n \leq b_n$ for all n, then which of the following statements MUST be true? Be sure to mark ALL correct answers.

   (a) If $\sum_{n=0}^{\infty} a_n$ converges then $\sum_{n=0}^{\infty} b_n$ converges.
   (b) If $\sum_{n=0}^{\infty} a_n$ diverges then $\sum_{n=0}^{\infty} b_n$ diverges.
   (c) If $\sum_{n=0}^{\infty} b_n$ converges then $\sum_{n=0}^{\infty} a_n$ converges.
   (d) If $\sum_{n=0}^{\infty} b_n$ diverges then $\sum_{n=0}^{\infty} a_n$ diverges.

5. Which of the following statements are true? Be sure to mark ALL correct answers.

(a) $\displaystyle\sum_{n=1}^{3} \frac{1}{\sqrt{n}} \geq \int_1^4 \frac{1}{\sqrt{n}}$ $\qquad$ (b) $\displaystyle\sum_{n=1}^{3} \frac{1}{\sqrt{n}} \leq \int_1^4 \frac{1}{\sqrt{n}}$

(c) $\displaystyle\sum_{n=2}^{4} \frac{1}{\sqrt{n}} \geq \int_1^4 \frac{1}{\sqrt{n}}$ $\qquad$ (d) $\displaystyle\sum_{n=2}^{4} \frac{1}{\sqrt{n}} \leq \int_1^4 \frac{1}{\sqrt{n}}$

6. The two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n (x-1)^n$ are both converge for all x and are equal. That is, for every x,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n (x-1)^n.$$

Which of the following must be true? Be sure to mark ALL correct answers.

$$(a)\quad a_0 = \sum_{n=0}^{\infty} (-1)^n b_n \qquad\qquad (b)\quad a_0 = \sum_{n=0}^{\infty} b_n$$

$$(c)\quad b_0 = \sum_{n=0}^{\infty} (-1)^n a_n \qquad\qquad (d)\quad b_0 = \sum_{n=0}^{\infty} a_n$$

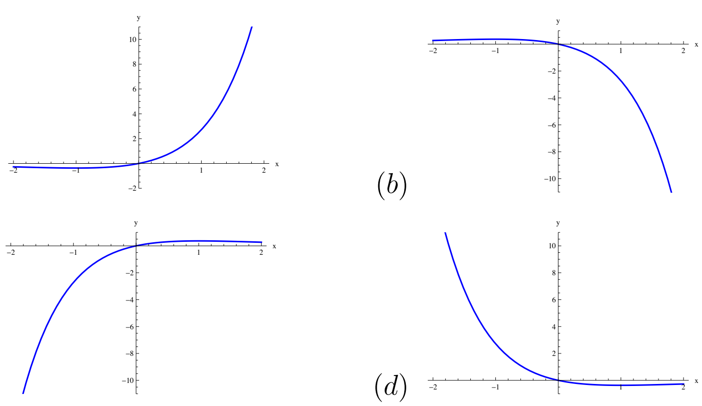
7. The series $\sum_{n=1}^{\infty} a_n$ has the property that its n-th partial sum $s_n = a_1 + a_2 + \cdots + a_n$ for every n is $\frac{1}{\sqrt{n}}$. Does the series converge and if so to what value?

8. Find the sum of the following infinite series:

$$1 - \frac{\pi^2}{2!2^2} + \frac{\pi^4}{4!2^4} - \frac{\pi^6}{6!2^6} + \frac{\pi^8}{8!2^8} + \ldots$$

9. The following power series converges for all x and defines a function $f(x)$. Which is the graph of the function determined by this power series?

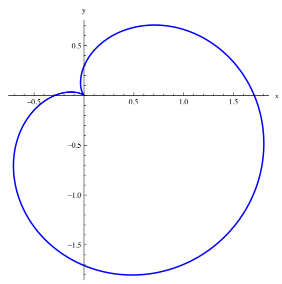
$$\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x + x^2 + \frac{x^3}{2\cancel{3}!} + \cdots\ ?$$



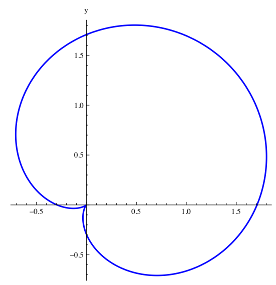
(a) (b) (c) (d)

10. Which of the following is the graph of

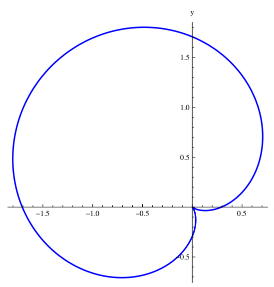
$$r = 1 + \cos\left(\theta + \frac{\pi}{4}\right)\ ?$$



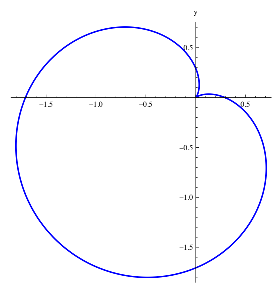
(a)



(b)

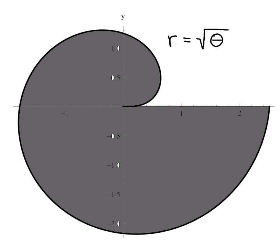


(c)



(d)

11. Determine the area of the shaded region determined by the graph of $r = \sqrt{\theta}$ in polar coordinates.



12. Find the length of the polar curve

$$r = \cos^3(\theta/3), \quad 0 \leq \theta \leq \pi/4.$$

13. Evaluate the following integrals:

   (a)

$$\int x \cos(x)\, dx$$

   (b)

$$\int \frac{x^3}{\sqrt{1 - x^2}}\, \mathrm{d}x.$$

(c)

$$\int \frac{x^2 + 2x}{(x+1)^2}\, \mathrm{d}x.$$

(d)

$$\int \sec^3(2\theta)\tan^3(2\theta)\, d\theta$$

14. Does the following improper integral converge? Justify your answer.

$$\int_0^2 \frac{\mathrm{d}t}{t^2 - 1}.$$

15. Find a series representation for the definite integral $\int_0^1 e^{3x^3} dx$.

16. Find the Taylor series (around $x = 0$) for $f(x) = \ln(4 + 3x^2)$. Give the radius of convergence. (You don't need to find the interval of convergence)

17. Compute the radius of convergence of the following power series. If the radius of convergence is finite also check the endpoints to determine if the series converges there to determine the interval of convergence.

$$\sum_{k=1}^{\infty} \frac{k!}{k^{2k}}\, x^k.$$

18. Investigate the following series for absolute convergence, conditional convergence or divergence:

(a)

$$\sum_{n=\cancel{1}\,2}^{\infty} \frac{1}{n\ln(n)}.$$

(b)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + \sqrt{k}}$$

(c)

$$\sum_{n=1}^{\infty} n\sin(\frac{1}{n}).$$

19. Give the definition of the Taylor series of a (sufficiently differentiable) function $f(x)$ around $x = c$ and use this to find the Taylor series of $f(x) = \frac{1}{x}$ around $x = 1$.

20. Recall that the decimal expansion for a number $a \in [0,1]$ is given as

$$a = .a_1 a_2 a_3 a_4 \ldots$$

where each a_i is some non-negative integer less than 10 and it means that

$$a = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$$

Use comparison with a geometric series to show that for ANY sequence of non-negative integers $\{a_n\}$ with $0 \leq a_n \leq 9$, the series $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges to a number at most 1.

Final Sample Exam (Solutions)

1) $(1-r)(1+r+r^2+\cdots+r^n) = 1+r+r^2+\cdots+r^n - r - r^2 - r^3 - \cdots$

$= 1-r^{n+1}$

$\Rightarrow 1+r+\cdots+r^n = \dfrac{1-r^{n+1}}{1-r}$

2) $\dfrac{1}{(x^2-1)(x^2+3)} = \dfrac{1}{(x-1)(x+1)(x^2+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+3}$

a

3) If $x>0$,

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \underbrace{\dfrac{x^9}{9!} - \dfrac{x^{11}}{11!}}_{>0} + \underbrace{\dfrac{x^{13}}{13!} - \dfrac{x^{15}}{15!}}_{>0} + \cdots$

$\geq x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \underbrace{\dfrac{x^9}{9!} - \dfrac{x^{11}}{11!}}_{>0}$

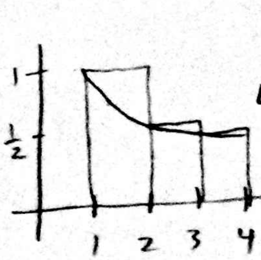
$> x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$

So $\boxed{\text{True}}$

4) By the comparison test, b and c are true

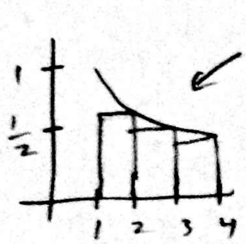
(a and d are false because, e.g., if $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n}$, $a_n < b_n$ but a and d are false)

5)



This shows $\sum_{n=1}^{3} \frac{1}{\sqrt{n}} > \int_1^4 \frac{dx}{\sqrt{x}}$

so a is true and b is false



This shows $\sum_{n=2}^{4} \frac{1}{\sqrt{n}} < \int_1^4 \frac{dx}{\sqrt{x}}$

so c is false and d is true.

6) Let $x=1$. Then the equation becomes

$$\sum_{n=0}^{\infty} a_n = b_0 \text{ , so (d) is true}$$

Let $x=0$. Then the equation becomes

$$a_0 = \sum_{n=0}^{\infty} (-1)^n b_n \text{ so (a) is true.}$$

7) By definition, $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{7n} = 0$

so $\sum_{n=1}^{\infty} a_n$ converges to 0.

8)

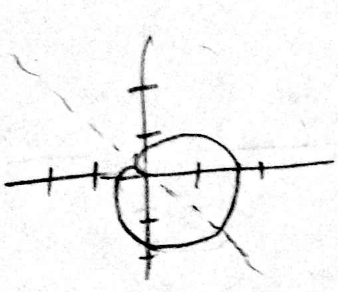
$$1 - \frac{\pi^2}{2!\,2^2} + \frac{\pi^4}{4!\,2^4} - \frac{\pi^6}{6!\,2^6} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}$$

$$= \cos(\pi/2) = \boxed{0}$$

9) $\sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = x\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} \overset{m=n-1}{=} x \sum_{m=0}^{\infty} \frac{x^m}{m!} = xe^x$, which goes quickly to infinity as $x \to \infty$. Its graph is (a).

10)

θ	r
0	$1+\sqrt{2}/2$
$\pi/4$	1
$\pi/2$	$1-\sqrt{2}/2$
$3\pi/4$	0
π	$1-\sqrt{2}/2$
$5\pi/4$	1
$3\pi/2$	$1+\sqrt{2}/2$
$7\pi/4$	2



so the answer is (a)

11) $\text{Area} = \int_0^{2\pi} \frac{1}{2}(\sqrt{\theta})^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\theta\, d\theta$

$= \frac{\theta^2}{4}\Big|_0^{2\pi} = \boxed{\pi^2}$

12) $\frac{dr}{d\theta} = 3\cos^2(\theta/3) \cdot (-\sin(\theta/3)) \cdot \frac{1}{3} = -\cos^2(\frac{\theta}{3})\sin(\frac{\theta}{3})$

$\text{Length} = \int_0^{\pi/4} \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta = \int_0^{\pi/4} \sqrt{\cos^6(\frac{\theta}{3}) + \cos^4(\frac{\theta}{3})\sin^2(\frac{\theta}{3})}\, d\theta$

$= \int_0^{\frac{\pi}{4}} \cos^2(\frac{\theta}{3})\sqrt{\cos^2(\frac{\theta}{3}) + \sin^2(\frac{\theta}{3})}\, d\theta$

$= \int_0^{\pi/4} \cos^2(\frac{\theta}{3})\, d\theta$

$= \int_0^{\pi/4} \frac{1}{2}(1 + \cos(\frac{2\theta}{3}))\, d\theta$

$= \frac{\theta}{2} + \frac{3\sin(\frac{2\theta}{3})}{4}\Big|_0^{\pi/4}$

$= \frac{\pi}{8} + \frac{3}{4}\sin(\frac{\pi}{6}) = \boxed{\frac{\pi}{8} + \frac{3}{8}}$

13) a) $u = x$ $\quad dv = \cos x\, dx$

$du = dx$ $\quad v = \sin x$

$\int x\cos x\, dx = x\sin x - \int \sin x\, dx = \boxed{x\sin x + \cos x + C}$

b) Method 1

$x = \sin\theta$

$dx = \cos\theta\, d\theta$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int \sin^3\theta\, d\theta$$

$$= \int \sin\theta(1-\cos^2\theta)\,d\theta$$

$$= \int \sin\theta\, d\theta - \int \sin\theta\cos^2\theta\, d\theta$$

$$= -\cos\theta + \int u^2\, du$$

$$= -\cos\theta + \frac{u^3}{3} + C$$

$$= -\cos\theta + \frac{\cos^3\theta}{3} + C$$

$$= \boxed{-\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} + C}$$

Method 2

$u = 1-x^2 \qquad x^2 = 1-u$

$du = -2x\,dx$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int -\frac{1}{2}\,\frac{(1-u)\,du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2} - u^{1/2}\,du$$

$$= -\frac{1}{2}\cdot\left(2u^{1/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= -u^{1/2} + \frac{1}{3}u^{3/2} + C$$

$$= \boxed{-\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C}$$

c) $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \frac{x^2+2x}{x^2+2x+1}\,dx = \int \frac{(x^2+2x+1)-1}{x^2+2x+1}\,dx$

$= \int 1 - \frac{1}{(x+1)^2}\,dx$

$= \boxed{x + \frac{1}{x+1} + C}$

d) $\int \sec^3(2\theta)\tan^3(2\theta)\,d\theta$

$= \int \sec^2(2\theta)(\sec^2(2\theta)-1)\sec 2\theta \tan 2\theta\, d\theta$

$= \int x^2(x^2-1)\frac{dx}{2}$

$\boxed{\begin{array}{l} x = \sec 2\theta \\ dx = 2\sec 2\theta \tan 2\theta\, d\theta \end{array}}$

$= \left(\frac{x^5}{5} - \frac{x^3}{3}\right)\frac{1}{2} + C$

$= \boxed{\frac{\sec^5(2\theta)}{10} - \frac{\sec^3(2\theta)}{6} + C}$

14) $\int_0^2 \frac{dt}{t^2-1} = \int_0^1 \frac{dt}{t^2-1} + \int_1^2 \frac{dt}{t^2-1}$

$= \lim_{b\to 1^-} \int_0^b \frac{dt}{t^2-1} + \lim_{a\to 1^+} \int_a^2 \frac{dt}{t^2-1}$

$= \lim_{b\to 1^-} \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Big|_0^b + \lim_{a\to 1^+} \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Big|_a^2$

$= (-\infty - 0) + \left(\frac{1}{2}\ln\left(\frac{1}{3}\right) + \infty\right)$ The improper integral diverges.

$\int \frac{dt}{t^2-1} = \int \frac{dt}{(t+1)(t-1)} = \int \frac{-1/2}{t+1} + \frac{1/2}{t-1}\,dt = \frac{1}{2}(\ln|t-1| - \ln|t+1|) + C$

$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$

$\frac{1}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} \Rightarrow 1 = A(t-1) + B(t+1)$

$= (A+B)t + (B-A) \Rightarrow B = \frac{1}{2},\ A = \frac{-1}{2}$

15) $\int_0^1 e^{3x^3}dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(3x^3)^n}{n!}dx$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{3^n x^{3n}}{n!}dx$$

$$= \sum_{n=0}^{\infty} \frac{3^n x^{3n+1}}{n!(3n+1)} \Big|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{n!(3n+1)}$$

16) $f(x)=\ln(4+3x^2)$ $\qquad \frac{df}{dx} = \frac{6x}{4+3x^2}$

$$\frac{1}{4+3x^2} = \frac{1}{4(1+\frac{3x^2}{4})} = \frac{1}{4(1-(\frac{-3x^2}{4}))} = \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{-3}{4}x^2\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\tfrac{3}{4}\right)^n\frac{x^{2n}}{4}$$

so $\frac{6x}{4+3x^2} = \sum_{n=0}^{\infty} 6(-1)^n\left(\tfrac{3}{4}\right)^n\frac{x^{2n+1}}{4}$

so

$$\ln(4+3x^2) = \int \frac{6x}{4+3x^2}dx = \int \sum_{n=0}^{\infty} 6(-1)^n\left(\tfrac{3}{4}\right)^n\frac{x^{2n+1}}{4}dx$$

$$= \sum_{n=0}^{\infty} \frac{6(-1)^n}{4}\left(\tfrac{3}{4}\right)^n\frac{x^{2n+2}}{2n+2} + C$$

$\underline{x=0}$: $\boxed{\ln(4) = C}$

$$= \boxed{\sum_{n=0}^{\infty} \frac{6(-1)^n}{4}\left(\tfrac{3}{4}\right)^n\frac{x^{2n+2}}{2n+2} + \ln(4)}$$

$$\lim_{n\to\infty}\left|\left(\tfrac{3}{4}\right)^{n+1}\frac{x^{2n+4}}{2n+4}\cdot\frac{2n+2}{x^{2n+2}}\left(\tfrac{3}{4}\right)^{-n}\right|$$

$$= \lim_{n\to\infty}\frac{3|x|^2}{4}\cdot\frac{2n+2}{2n+4} = \frac{3}{4}(x)^2 < 1 \Rightarrow |x| < \frac{2}{\sqrt{3}}$$

$$\boxed{R = 2/\sqrt{3}}$$

17) $\sum_{k=1}^{\infty} \frac{k!}{k^{2k}} x^k$

$$\lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{(k+1)^{2(k+1)}} \cdot \frac{k^{2k}}{k!\, x^k} \right| = \lim_{k\to\infty} \frac{(k+1)\, k^{2k}}{(k+1)^{2(k+1)}} |x|$$

$$= \lim_{k\to\infty} \frac{k^{2k}}{(k+1)^{2k+1}} |x|$$

$$= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^{2k} \frac{|x|}{k+1}$$

$$\leq \lim_{k\to\infty} \frac{|x|}{k+1} = 0 < 1$$

for all x

$\boxed{\begin{array}{l}\text{radius} = \infty \\ \text{interval}: (-\infty, \infty)\end{array}}$

18) a) $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

$u = \ln x$

$du = 1/x \, dx$

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{1}{u} du = \lim_{t\to\infty} \ln|t| - \ln 2 = \infty$$

$\boxed{\text{diverges}}$

b) $\sum_{k=1}^{\infty} \frac{(-1)^k}{k+\sqrt{k}}$ $\boxed{\text{converges conditionally}}$

- The sequence $\left(\frac{1}{k+\sqrt{k}}\right)$ is decreasing since

$\frac{d}{dk} \frac{1}{k+\sqrt{k}} = -(k+\sqrt{k})^{-2}\left(1+\frac{1}{2\sqrt{k}}\right) < 0$ and goes to 0 as $k\to\infty$.

By the A.S.T, the series converges.

- $\frac{1}{k+\sqrt{k}} > \frac{1}{k+k} = \frac{1}{2k}$. Since $\sum \frac{1}{2k}$ diverges by p-series ($p=1$)

$\sum 1/k+\sqrt{k}$ diverges by the C.T.

1c) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \neq 0$$

So the given series diverges by the n^{th} term test.

19) If f has derivatives of all orders at $x=c$, the Taylor series of f at $x=c$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$f(x) = \frac{1}{x}$ $\qquad f(1) = 1$

$f'(x) = -x^{-2}$ $\qquad f'(1) = -1$

$f''(x) = 2x^{-3}$ $\qquad f''(1) = 2$

$f'''(x) = -3 \cdot 2x^{-4}$ $\qquad f'''(1) = -6$

$f^{(4)}(x) = 4 \cdot 3 \cdot 2 x^{-5}$ $\qquad f''''(1) = 24$

$\vdots$ $\qquad \vdots$

$f^{(n)}(x) = (-1)^n n!\, x^{-(n+1)}$ $\qquad f^{(n)}(1) = (-1)^n n!$

Taylor series: $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$

20) Let (a_n) be a sequence of nonnegative integers with $0 \le a_n \le 9$ for all n. Then

$$0 \le \sum_{n=1}^{\infty} \frac{a_n}{10^n} \le \sum_{n=1}^{\infty} \frac{9}{10^n} = 9 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 9 \cdot \frac{1/10}{1-1/10}$$

$$= 1$$

So $\sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges, and its sum is at most 1.