Math231 Exam 1
2013

1. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int x\cos(3x)\,dx$

   (b) $\displaystyle\int \sec^4(5x)\,dx$

   (c) $\displaystyle\int \frac{3\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha$

2. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int \frac{dx}{(25+x^2)^{\frac{3}{2}}}$

   (b) $\displaystyle\int \frac{x^2}{x^2+9}\,dx$

   (c) $\displaystyle\int \frac{x+a}{x^2-x}\,dx$

3. (*10 points*) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

   (a) $\displaystyle\int_1^\infty \frac{dx}{\sqrt[3]{x}}$

   (b) $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

4. (*8 points*)

   For any twice differentiable function f on $[a,b]$, $P_a^b(f)$ will approximate $\int_a^b f(x)\,dx$ to an error no more than $K_2\frac{(b-a)^4}{32}$ when $|f''(x)| \le K_2$ for all x in $[a,b]$. You use $P(f)$ to numerically approximate the integral $\displaystyle\int_1^3 \sqrt{1+x^3}\,dx$ by subdividing the interval into 10 equal pieces and applying $P(f)$ to each of the smaller intervals.

Using that the absolute value of the second derivative of $\sqrt{1+x^3}$ is never more than 2 over $[1,3]$, what is an upper bound for the error of your approximation (*4 points*) and why (*6 points*)?

5. (*12 points, 8/4*) Let $a > 0$. Evaluate the integrals.

   (a) $\displaystyle\int \sin(x)e^{-ax}\,dx$

   (b) $\displaystyle\int_0^{\infty} \sin(x)e^{-ax}\,dx$

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1 a) $\int x\cdot\cos(3x)\,dx$ $\qquad u = x \quad du = dx \qquad dv = \cos(3x) \quad v = \frac{1}{3}\sin(3x)$

$$= \frac{1}{3}x\cdot\sin(3x) - \frac{1}{3}\int \sin 3x\,dx$$

$$= \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C$$

b) $\int \sec^4(5x)\,dx = \int \sec^2(5x)\cdot\sec^2(5x)\,dx = \int(\tan^2(5x)+1)\cdot\sec^2(5x)\,dx$

$u = \tan(5x)$
$du = 5\cdot\sec^2(5x)dx$

$$= \underbrace{\int \tan^2(5x)\cdot\sec^2(5x)\,dx} + \int\sec^2(5x)\,dx$$

$$= \frac{1}{5}\int u^2\,du + \frac{1}{5}\tan(5x)$$

$$= \frac{1}{15}u^3 + \frac{1}{5}\tan(5x) + C$$

$$= \frac{1}{15}\tan^3(5x) + \frac{1}{5}\tan(5x) + C$$

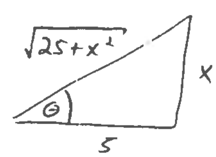
c) $\int \frac{3\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha = 3\int\frac{\cos^4\alpha\cdot\cos\alpha}{\sqrt{\sin\alpha}}\,d\alpha = 3\int\frac{(1-\sin^2\alpha)^2}{\sqrt{\sin\alpha}}\cos\alpha\,d\alpha$ $\qquad u = \sin\alpha \quad du = \cos\alpha\,d\alpha$

$$= 3\int\frac{(1-u^2)^2}{\sqrt{u}}\,du = 3\int\frac{1-2u^2+u^4}{\sqrt{u}}\,du = 3\int\left(u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}\right)du$$

$$= 3\left(2\cdot u^{\frac{1}{2}} - 2\cdot\frac{2}{5}\cdot u^{\frac{5}{2}} + \frac{2}{9}\cdot u^{\frac{9}{2}}\right) + C = 6\sqrt{u} - \frac{12}{5}u^2\sqrt{u} + \frac{2}{3}u^4\sqrt{u} + C$$

$$= \sqrt{\sin\alpha}\cdot\left(6 - \frac{12}{5}\sin^2\alpha + \frac{2}{3}\sin^4\alpha\right) + C$$

2a) $\int\frac{dx}{(25+x^2)^{3/2}} = \int\frac{5\cdot\sec^2\theta\,d\theta}{5^3\cdot\sec^3\theta}$ $\qquad x = 5\cdot\tan\theta \quad dx = 5\cdot\sec^2\theta\,d\theta$



$$= \frac{1}{5^2}\int\frac{1}{\sec\theta}\,d\theta = \frac{1}{5^2}\int\cos\theta\,d\theta$$

$$= \frac{1}{25}\sin\theta + C = \frac{1}{25}\frac{x}{\sqrt{x^2+25}} + C$$

$\cos\theta = \frac{5}{\sqrt{x^2+25}}$

$\sec\theta = \frac{\sqrt{x^2+25}}{5}$

$\sec^3\theta = \frac{(x^2+25)^{3/2}}{5^3}$

$\sin\theta = \frac{x}{\sqrt{25+x^2}}$

b) $\int\frac{x^2}{x^2+9}\,dx = \int\frac{x^2+9-9}{x^2+9}\,dx = \int\left(1 - \frac{9}{x^2+9}\right)dx = \int 1\,dx - \int\frac{1}{\left(\frac{x}{3}\right)^2+1}\,dx$

$$= x - 3\cdot\arctan\left(\frac{x}{3}\right) + C$$

c) $\int\frac{x+a}{x^2-x}\,dx = \int\frac{x+a}{x(x-1)}\,dx = \int\left(\frac{A}{x} + \frac{B}{x-1}\right)dx$

$Ax - A + Bx = x + a$
$A + B = 1$
$-A = a \rightarrow \boxed{A = -a \quad B = 1+a}$

$$= -\int\frac{a}{x}\,dx + \int\frac{1+a}{x-1}\,dx = \underline{-a\ln|x| + (1+a)\cdot\ln|x-1| + C}$$

3) a) $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt[3]{x}} = \int_1^{\infty} \frac{dx}{x^{\frac{1}{3}}}$

By the p-test, $\displaystyle\int_1^{\infty} \frac{dx}{x^p}$ converges iff $p > 1$ so this integral is divergent

b) $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}} = \int_0^1 \frac{dx}{x^{\frac{1}{3}}}$

By the p-test, $\displaystyle\int_0^1 \frac{dx}{x^p}$ converges iff $p < 1$ so this integral is convergent

$$= \left[\frac{3}{2} x^{\frac{2}{3}}\right]_0^1$$

$$= \frac{3}{2}(1 - 0) = \frac{3}{2}$$

4) $\displaystyle\int_1^3 \sqrt{1+x^3}\, dx$ approximated by subdividing into 10 equal pieces $\longrightarrow \Delta x = \frac{2}{10}$

The error for each subinterval is bounded by $K_2 \cdot \dfrac{\left(\frac{2}{10}\right)^4}{32}$

Summing over all 10 subintervals: $K_2 \cdot \dfrac{\left(\frac{2}{10}\right)^4}{32} \cdot 10 = \dfrac{K_2 \cdot 2^4 \cdot 10}{10^4 \cdot 2^5} = \dfrac{K_2}{2000}$

Reason: $\displaystyle\left|\int_a^b f(x)dx - P[a,b]\right| = \left|\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx - \sum_{i=1}^{n} P[x_{i-1}, x_i]\right|$

$$\le \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(x)dx - P[x_{i-1}, x_i]\right| \quad \text{by triangle inequality}$$

$$\le \sum_{i=1}^{n} \frac{K_2 \cdot \left(\frac{b-a}{n}\right)^4}{32} = \frac{n \cdot K_2 \cdot \left(\frac{b-a}{n}\right)^4}{32}$$

$n = 10$
$b = 3$
$a = 1$

$$\longrightarrow \frac{10 \cdot K_2 \cdot \left(\frac{3-1}{10}\right)^4}{32} = \frac{K_2 \cdot 10 \cdot 2^4}{10^4 \cdot 2^5} = \frac{K_2}{2000}$$

5) a) $\displaystyle\int \sin(x) \cdot e^{-ax}\, dx$ for $a > 0$

$u_1 = \sin x \qquad dv_1 = e^{-ax} dx$
$du_1 = \cos x\, dx \qquad v_1 = -\frac{1}{a} e^{-ax}$
$u_2 = \cos x \qquad dv_2 = e^{-ax} dx$
$du_2 = -\sin x\, dx \qquad v_2 = -\frac{1}{a} e^{-ax}$

$$= -\frac{1}{a} \sin x \cdot e^{-ax} + \frac{1}{a} \int e^{-ax} \cos x\, dx$$

$$= -\frac{1}{a} \sin x \cdot e^{-ax} + \frac{1}{a} \cdot \left(-\frac{1}{a} \cos x\, e^{-ax} - \frac{1}{a} \int \sin x\, e^{-ax}\, dx\right)$$

$$= -\frac{1}{a} \sin x \cdot e^{-ax} - \frac{1}{a^2} \cdot \cos x\, e^{-ax} - \frac{1}{a^2} \int \sin x \cdot e^{-ax} dx$$

$$= \frac{a^2}{a^2+1} \cdot \left(-\frac{1}{a} \sin x \cdot e^{-ax} - \frac{1}{a^2} \cos x \cdot e^{-ax}\right) + C$$

b) $\displaystyle\int_0^{\infty} \sin x\, e^{-ax} dx = \lim_{c \to \infty} \int_0^c \sin x\, e^{-ax} dx$

$$= \lim_{c \to \infty} \left[\frac{a^2}{a^2+1} \cdot \left(-\frac{1}{a} \sin x\, e^{-ax} - \frac{1}{a^2} \cdot \cos x\, e^{-ax}\right)\right]_0^c$$

$$= \lim_{c \to \infty} \frac{a^2}{a^2+1} \cdot \left[\left(\underbrace{-\frac{1}{a} \sin c\, e^{-ac}}_{\to 0} - \underbrace{\frac{1}{a^2} \cdot \cos c\, e^{-ac}}_{\to 0}\right) - \left(\underbrace{-\frac{1}{a} \sin 0 \cdot e^{-a \cdot 0}}_{=0} - \underbrace{\frac{1}{a^2} \cdot \cos 0 \cdot e^{-a0}}_{= -\frac{1}{a^2}}\right)\right]$$

$$= \lim_{c \to \infty} \frac{a^2}{a^2+1} \cdot \frac{1}{a^2} = \frac{1}{a^2+1} \text{ for } a > 0$$