Math 231 Exam III

UIUC, April 18, 2013

1. (*8 points*) Short answer.

   (a) Suppose that $c(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$ but diverges for $x = 6$.

      i. $\sum_{n=0}^{\infty} 2^n c_n$ (absolutely converges/ conditionally converges/diverges).

      Solution: Since $c(x)$ is centered at 0 and converges at -4 the series absolutely converges on $(-4, 4)$ and in particular at 2, so $\sum_{n=0}^{\infty} 2^n c_n$ absolutely converges.

      ii. $\sum_{n=0}^{\infty}(-8)^n c_n$ (absolutely converges/ conditionally converges/diverges).

      Solution: Since $c(x)$ is centered at 0 and diverges at 6 the series diverges for all $|x| > 6$ and in particular at -8, so $\sum_{n=0}^{\infty}(-8)^n c_n$ diverges.

   (b) Calculate the binomial coefficient $\binom{-3}{3} =$

   Solution:

$$\binom{-3}{3} = \frac{-3(-3-1)(-3-2)}{3!} = \frac{-3(-4)(-5)}{6}$$

   (c) Recall that $\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$. By the Alternating Series Estimation, how accurate is $1 - \frac{1}{2} + \cdots - \frac{1}{8} + \frac{1}{9} = \frac{1879}{2520}$ to the actual value of $\ln 2$?

   Solution: The error is given by the next term of the sequence, which is $-\frac{1}{10}$ and so this is an over estimate to $\ln 2$ by no more than $\frac{1}{10}$. (Note, $\ln 2 = .693....$ while $\frac{1879}{2520} = .745...$)

2. (*10 points*) Find a series solution to the integral

$$\int_0^1 e^{-x^2}\,dx$$

Solution: Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with infinite radius of convergence,

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^1 x^{2n}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2n+1}$$

3. (*12 points each*) Find the radius and interval of convergence for the power series. Be sure to indicate which points converge absolutely and which converge conditionally.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2 \cdot 4 \cdot 6 \cdots (2n+2)}$

Solution: By Ratio Test

$$\lim_{n\to\infty} \frac{\left|\frac{(x+2)^{n+1}}{2\cdot 4\cdots(2n+2)(2(n+1)+2)}\right|}{\left|\frac{(x+2)^n}{2\cdot 4\cdots(2n+2)}\right|} = |x+2| \lim_{n\to\infty} \frac{2\cdot 4\cdots(2n+2)}{2\cdot 4\cdots(2n+2)(2n+4)} = |x+2| \lim_{n\to\infty} \frac{1}{2n+4} = 0$$

Thus, the series absolutely converges for all x and so the radius is ∞ and the interval is $(-\infty, \infty)$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x+2)^n}{(n+1)(n)}$

Solution: By Ratio Test

$$\lim_{n\to\infty} \frac{\left|\frac{(3x+2)^{n+1}}{((n+1)+1)(n+1)}\right|}{\left|\frac{(3x+2)^n}{(n+1)(n)}\right|} = |3x+2| \lim_{n\to\infty} \frac{(n+1)(n)}{(n+2)(n+1)} = |3x+2| \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{3}{n}+\frac{2}{n^2}} = |3x+2|$$

Thus, the series converges absolutely when $|3x+2| < 1$ or $|x+\frac{2}{3}| < \frac{1}{3}$. The Radius of the series is $\frac{1}{3}$ (centered at $-\frac{2}{3}$) and the endpoints of the interval are when $|3x+2| = 1$ or $x = -1$ and $x = -\frac{1}{3}$.
When $x = -\frac{1}{3}$ we obtain

$$\sum_{n=0}^{\infty} \frac{(3(-1/3)+2)^n}{(n+1)(n)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)n} < \sum_{n=0}^{\infty} \frac{1}{n^2}$$

and the series absolutely converges by comparison with a p-series for $p = 2$.
When $x = -1$ we obtain

$$\sum_{n=0}^{\infty} \frac{(3(-1)+2)^n}{(n+1)(n)} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)n}$$

which also absolutely converges by the previous case.
Thus, the interval of convergence is [-1,1] with absolute convergence at the endpoints.

4. (*12 points*) Give the Taylor polynomial with degree 2 centered at 1 for $f(x) = \sqrt[5]{x}$. Then use Taylor's Inequality to estimate the accuracy of this approximation for x between 1 and 1.1.

Solution: We begin by tabulating the derivatives of f.

Derivative	Function	at $a = 1$
f	$x^{\frac{1}{5}}$	1
f'	$\frac{1}{5}x^{-\frac{4}{5}}$	$\frac{1}{5}$
f''	$\frac{1}{5}(-\frac{4}{5})x^{\frac{-9}{5}}$	$-\frac{4}{25}$
$f^{(3)}$	$\frac{1}{5}(-\frac{4}{5})(-\frac{9}{5})x^{-\frac{14}{5}}$	

So the Taylor polynomial with degree 2 at 1 for $f(x) = x^{\frac{1}{5}}$ is

$$f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} = 1 + \frac{1}{5}(x-1) - \frac{2}{25}(x-1)^2$$

By Taylor's Remainder Theorem

$$\sqrt[5]{x} = 1 + \frac{1}{5}(x-1) - \frac{2}{25}(x-1)^2 + f^{(3)}(c)\frac{(x-1)^3}{3!}$$

for some c between 1 and x. Thus, for x between 1 and 1.1, c is also between 1 and 1.1. Since $f^{(3)} = \frac{36}{125}\frac{1}{\sqrt[5]{x^{14}}}$ is decreasing from 1 to 1.1 (the bigger the denominator the smaller the fraction) the largest $f^{(3)}(c)$ can be is if $c = 1$. The largest $|x-1|^3$ can be is when $x = 1.1$. Thus the error of the estimation from 1 to 1.1 is bounded by

$$|\text{error}| = \left|f^{(3)}(c)\frac{(x-1)^3}{3!}\right| \leq \left|f^{(3)}(1)\frac{(1.1-1)^3}{3!}\right| = \left|\frac{36}{125}\frac{1}{\sqrt[5]{1^{14}}}\right|\frac{|1.1-1|^3}{3!} = \frac{6}{125 \cdot 10^3}$$

5. (*12 points*) Let f be a function which has all derivatives and has the property that $f'' = f$. If $f(0) = 0$ and $f'(0) = 1$, what is the power series for f at 0?

Solution: We begin by tabulating the derivatives of f using the fact that

$$f^{(3)} = (f'')' = (f)' = f' \quad \text{and} \quad f^{(4)} = (f'')'' = (f)'' = f'' = f$$

Derivative	at $a =$
f	0
f'	1
f''	0
$f^{(3)}$	1
$f^{(4)} = f$	0

And so we see that $f^{(n)}(0)$ is 0 when n is even and 1 when n is odd. Since a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has the property that $c_n = \frac{f^{(n)}}{n!}$ we have that

$$f(x) \approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

(Note, $f(x) = \sinh x$).

6. (*12 points*) Determine a power series centered at 0 for $f(x) = \sin^{-1} x$ and use it to determine the 100-th derivative of $\sin^{-1} x$ at 0. You may find the following useful

$$\binom{-\frac{1}{2}}{n} = (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!}$$

Solution: Since $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} = (1 + (-x^2))^{\frac{1}{2}}$ we will integrate the series obtained by the Binomial Theorem to obtain our series for $\sin^{-1} x$. Recall,

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n; \qquad |x| < 1$$

and so

$$(1 + (-x^2))^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n; \qquad |x^2| < 1$$

Using the given formula and that $(-1)^n(-1)^n = 1$ this becomes

$$(1 + (-x^2))^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!} x^{2n}; \qquad |x| < 1$$

and so

$$\begin{aligned}
\sin^{-1} x &= C + \int (1 + (-x^2))^{\frac{1}{2}}\, dx \\
&= C + \int \left(\sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!} x^{2n} \right) dx \\
&= C + \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!} \int x^{2n}\, dx \\
&= C + \sum_{n=0}^{\infty} \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!} \frac{x^{2n+1}}{2n+1}
\end{aligned}$$

To determine C, we evaluate at $x = 0$ to obtain $0 = \sin^{-1} 0 = C + 0$ and so $C = 0$.

If $\sin^{-1} x = \sum_{n=0}^{\infty} c_n x^n$ then c_n is equal to the n-th derivative of $\sin^{-1} x$ at 0 divided by $n!$. In particular, the 100-th derivative of $\sin^{-1} x$ at 0 is 100! times the coefficient for x^{100} in our series. From our formula, the series has non-zero terms only at odd coefficients (x^{2n+1}) which means that $c_{100} = 0$ (since 100 is even) and so the 100-th derivative of $\sin^{-1} x$ at 0 is 0.