# Math 231/EL1 Final

UIUC, May 7, 2013

| Q. | Pt | Score | Q. | Pt | Score | Q. | Pt | Score | Q. | Pt | Score |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 16 | | 6 | 22 | | 11 | 8 | | 16 | 9 | |
| 2 | 16 | | 7 | 22 | | 12 | 10 | | 17 | 4 | |
| 3 | 16 | | 8 | 5 | | 13 | 20 | | 18 | 5 | |
| 4 | 5 | | 9 | 3 | | 14 | 10 | | TA | 2 | |
| 5 | 7 | | 10 | 8 | | 15 | 12 | | ExCr | 10 | |
| Tot | 60 | | Tot | 60 | | Tot | 60 | | Tot | 30 | |

1. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int x\sin(3x)\,dx$

   (b) $\displaystyle\int \sec^4(5x)\,dx$

2. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int \frac{3\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha$

   (b) $\displaystyle\int \frac{dx}{(25+x^2)^{\frac{3}{2}}}$

3. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int \frac{x^2}{x^2+4}$

   (b) $\displaystyle\int \frac{x+a}{x^2-x}\,dx$

4. (*5 points*) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

   $\displaystyle\int_{-1}^{2} \frac{dx}{x^{11}}$

5. (*7 points*) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^\infty xe^{-3x}\,dx$$

6. (*11 points each*)

Determine if the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to show your reasoning. No work, no credit.

(a) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{\sqrt{n^3+30n}}$

(b) $\sum_{n=2}^{\infty}(-1)^n \frac{n+3}{n}$

7. (*11 points each*)

Determine if the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to show your reasoning. No work, no credit.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{7^n}$

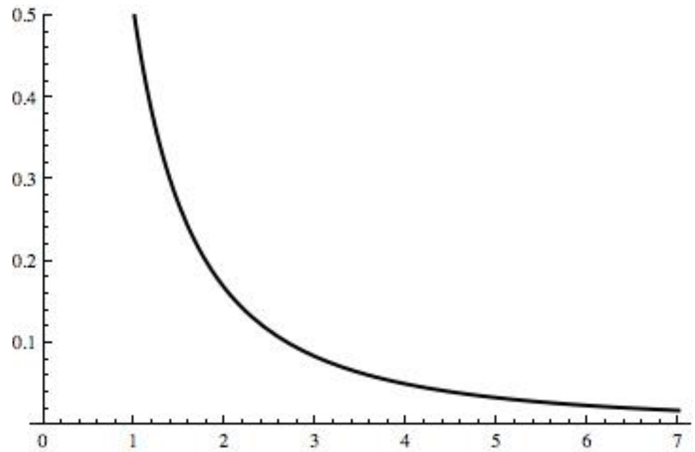
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n^n)}$

8. (*5 points*) Show that for any number $r \neq 1$ and positive integer $k$,

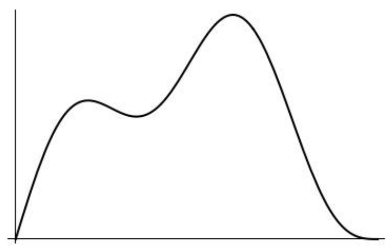
$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

9. (*3 points*) Draw on the diagram and give a brief explanation why

$$\sum_{n=2}^{6} \frac{1}{n(n+1)} \leq \int_1^6 \frac{dx}{x(x+1)}$$




$$f(x) = \frac{1}{x(x+1)}$$



10. (*8 points, 2/3/3*)

This problem concerns the curve

$$y = 2\sin x + \sin 2x, \qquad 0 \le x \le \pi$$

(a) Give an integral for the length of the curve. You do not need to evaluate the integral.

(b) Give an integral for the area of the surface obtained by rotating the curve about the $x$-axis. You do not need to evaluate the integral.

(c) Give an integral for the area of the surface obtained by rotating the curve about the $y$-axis. You do not need to evaluate the integral.

11. (*8 points*) Short answer.

(a) Suppose that $c(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$ but diverges for $x = 6$.

i. $\sum_{n=0}^{\infty}(-1)^n c_n$ (absolutely converges/ conditionally converges/diverges).

ii. $\sum_{n=0}^{\infty}(-7)^n c_n$ (absolutely converges/ conditionally converges/diverges).

(b) Calculate the binomial coefficient $\binom{-3}{4} =$

(c) Recall that $\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. By the Alternating Series Estimation, how accurate is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = \frac{263}{315}$ to the actual value of $\frac{\pi}{4}$?

12. (*10 points*) Find a series solution to the integral

$$\int_{-1}^{1} \sin(x^2)\, dx$$

13. (*10 points each*) Find the radius and interval of convergence for the power series. Be sure to indicate which points converge absolutely and which converge conditionally.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n}$

14. (*10 points*) Give the Taylor polynomial with degree 2 centered at 1 for $f(x) = \sqrt[3]{x}$. Then use Taylor's Inequality to estimate the accuracy of this approximation for $\sqrt[3]{2}$.

15. (*12 points*) Determine a power series centered at 0 for

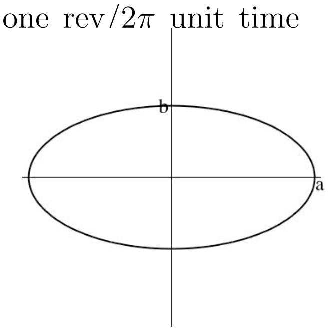
$$f(x) = \sin^{-1}(x^2)$$

and use it to determine the 104-th derivative of $\sin^{-1} x$ at 0. You may find the following useful

$$\binom{-\frac{1}{2}}{n} = (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!}$$

16. (*9 points, 3 points each*) Recall that the follow parametric equations model a particle traveling counter-clockwise about an ellipse at one rev$/2\pi$ unit time starting at the point $(a, 0)$.

$$x = a\cos t \qquad y = b\sin t$$



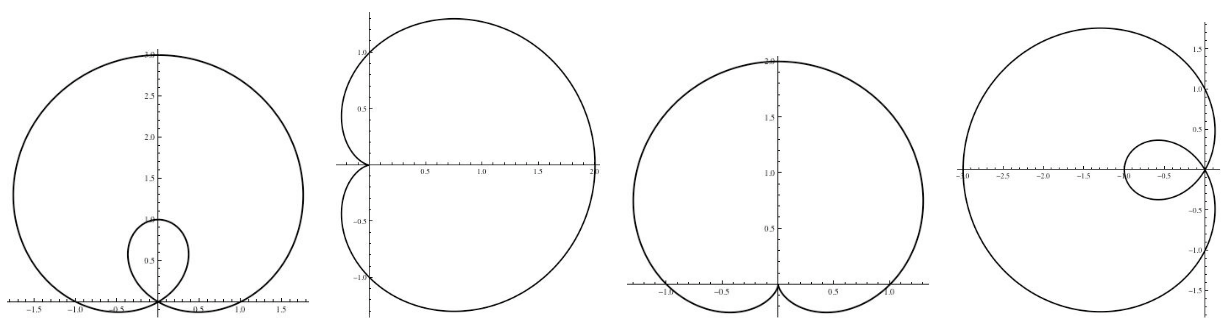

(a) What is the slope of the tangent line when $t = \frac{\pi}{4}$?

(b) Give an integral for the area of the ellipse using this set of parametric equations. You do not need to solve the integral.

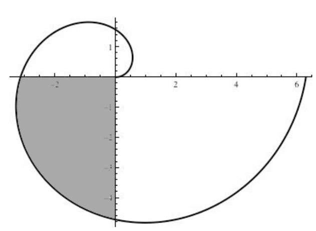
(c) Give an integral for the circumference of the ellipse using this set of parametric equations. You do not need to solve the integral.

17. (*4 points*) Match the graphs to their corresponding polar equations.

$$r = \sin\theta \quad r = \cos\theta \quad r = 1 + 2\sin\theta \quad r = 1 - 2\cos\theta$$



18. (*5 points*) Find the area of the shaded region of $r = e^{-\theta}$.



**Extra Credit** (*10 points*)

Recall that hyperbolic cosine is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

If we let the imaginary number $i = \sqrt{-1}$, use power series to establish the identity

$$\cos(ix) = \cosh(x)$$