Math 231/EL1 Final

UIUC, May 7, 2013

1. Evaluate the integral.

   (a) $\displaystyle\int x\sin(3x)\,dx$

   Solution: We will use integration by parts with $u = x$ and $dv = \sin(3x)\,dx$ and so $du = dx$ and $v = -\frac{1}{3}\cos(3x)$:

$$\int x\sin(3x)\,dx = -\frac{x}{3}\cos(3x) + \frac{1}{3}\int\cos(3x)\,dx = -\frac{x}{3}\cos(3x) + \frac{1}{9}\sin(3x) + C$$

   (b) $\displaystyle\int \sec^4(5x)\,dx$

   Solution: We use the trigonometric identity $\tan^2\theta + 1 = \sec^2\theta$ and the u substitution $u = \tan(5x)$:

$$\begin{aligned}\int \sec^4(5x)\,dx &= \int \sec^2(5x)\,\sec^2(5x)\,dx \\ &= \int \left(\tan^2(5x) + 1\right)\sec^2(5x)\,dx \\ &= \int \left(u^2 + 1\right)\frac{1}{5}\,du \\ &= \frac{1}{5}\left(\frac{u^3}{3} + u\right) + C \\ &= \frac{1}{5}\left(\frac{1}{3}\tan^3(5x) + \tan(5x)\right) + C\end{aligned}$$

2. (*8 points each*) Evaluate the integral.

   (a) $\displaystyle\int \frac{3\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha$

   Solution: We wish to use the u substitution $u = \sin\alpha$ and to aid us in this we factor out a copy of $\cos\alpha$ and use the trigonometric identity

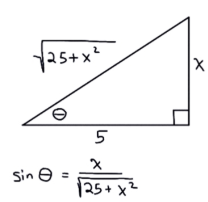
$\cos^2\theta = 1 - \sin^2\theta$ to rewrite:

$$\begin{aligned}
\int \frac{3\cos^5\alpha}{\sqrt{\sin\alpha}}\,d\alpha &= 3\int \frac{(1-\sin^2\alpha)^2}{\sqrt{\sin\alpha}}\ \cos\alpha\,d\alpha \\
&= 3\int \frac{(1-u^2)^2}{u^{\frac{1}{2}}}\,du \\
&= 3\int u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}}\,du \\
&= 3\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - 2\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{9}{2}}}{\frac{9}{2}}\right) + C \\
&= 6(\sin\alpha)^{\frac{1}{2}} - \frac{24}{5}(\sin\alpha)^{\frac{5}{2}} + \frac{2}{3}(\sin\alpha)^{\frac{9}{2}} + C
\end{aligned}$$

(b) $\displaystyle\int \frac{dx}{(25+x^2)^{\frac{3}{2}}}$

Solution: We use the trigonometric substitution $x = 5\tan\theta$:

$$\begin{aligned}
\int \frac{dx}{(25+x^2)^{\frac{3}{2}}}\,dx &= \int \frac{5\sec^2\theta}{(25+25\tan^2\theta)^{\frac{3}{2}}}\,d\theta \\
&= \frac{5}{5^3}\int \frac{\sec^2\theta}{(\sqrt{1+\tan^2\theta})^3}\,d\theta \\
&= \frac{1}{25}\int \frac{d\theta}{\sec\theta} = \frac{1}{25}\int \cos\theta\,d\theta \\
&= \frac{1}{25}\sin\theta + C
\end{aligned}$$



$$= \frac{1}{25}\left(\frac{x}{\sqrt{25+x^2}}\right) + C$$

3. (*8 points each*) Evaluate the integral.

(a) $\displaystyle\int \frac{x^2}{x^2+4}\,dx$

Solution: We begin with long division:

$$\frac{x^2}{x^2+4} = 1 + \frac{-4}{x^2+4}$$

and then integrate with the aid of the u substitution $u = \frac{x}{2}$:

$$\begin{aligned}\int \frac{x^2}{x^2+4}\,dx &= \int dx - 4\int \frac{dx}{x^2+4} \\ &= x - \int \frac{dx}{\left(\frac{x}{2}\right)^2+1} \\ &= x - 2\tan^{-1}\left(\frac{x}{2}\right) + C\end{aligned}$$

(b) $\displaystyle\int \frac{x+a}{x^2-x}\,dx$

Solution: We use a partial fraction decomposition from $x^2 - x = x(x-1)$:

$$\frac{x+a}{x^2-x} = \frac{-a}{x} + \frac{1+a}{x-1}$$

$$\int \frac{x+a}{x^2-x}\,dx = -a\int \frac{dx}{x} + (1+a)\int \frac{dx}{x-1} = -a\ln|x| + (1+a)\ln|x-1| + C$$

4. (*5 points*) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$\displaystyle\int_{-1}^{2} \frac{dx}{x^{11}}$

Solution: The integral is improper at 0.

$$\begin{aligned}\int_{-1}^{2} \frac{dx}{x^{11}} &= \lim_{b\to 0^-} \int_{-1}^{b} x^{-11}\,dx + \lim_{a\to 0^+} \int_{a}^{2} x^{-11}\,dx \\ &= \lim_{b\to 0^-} \frac{1}{b^{10}} + 1 + \lim_{a\to 0^+} \frac{1}{2^{10}} - \frac{1}{a^{10}}\end{aligned}$$

Neither of these limits exits ($\lim_{b\to 0^-} \frac{1}{b^{10}} = \infty$ while $\lim_{a\to 0^+} -\frac{1}{a^{10}} = -\infty$) and so the entire integral is divergent.

5. (*7 points*) Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

   $$\int_0^\infty xe^{-3x}\,dx$$

   Solution: The integral is only improper at ∞. We can integrate this using the method of parts with $u = x$ and $dv = e^{-3x}\,dx$ and so $v = -\frac{1}{3}e^{-3x}$:

   $$\begin{aligned}\int xe^{-3x}\,dx &= -\frac{x}{3}e^{-3x} + \frac{1}{3}\int e^{-3x}\,dx \\ &= -\frac{x}{3}e^{-3x} - \frac{1}{9}e^{-3x} + C\end{aligned}$$

   $$\begin{aligned}\int_0^\infty xe^{-3x}\,dx &= \lim_{b\to\infty}\int_0^b xe^{-3x}\,dx \\ &= \lim_{b\to\infty} -\frac{b}{3e^{3b}} - \frac{1}{9e^{3b}} + \frac{1}{9}\end{aligned}$$

   Since $e^x \to \infty$ as $x \to \infty$ we easily see that $\lim_{b\to\infty}\frac{1}{9e^{3b}} = 0$ and we can use L'Hôpital's rule to the indefinite form $\frac{\infty}{\infty}$ to determine

   $$\lim_{b\to\infty}\frac{b}{3e^{3b}} = \lim_{b\to\infty}\frac{1}{9e^{3b}} = 0$$

   and so the improper integral evaluates to $\frac{1}{9}$.

6. (*11 points each*)

   Determine if the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to show your reasoning. No work, no credit.

   (a) $\displaystyle\sum_{n=5}^{\infty}\frac{1}{\sqrt{n^3+30n}}$

   Solution: Since all the terms are positive, the series can only absolutely converge or diverge.

For large values of n, this integral will tend to behave like $\frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}$ which is a convergent p-series since $\frac{3}{2} > 1$. More specifically, we can use a limit comparison test with $\frac{1}{n^{\frac{3}{2}}}$ or easier still, a simple comparison test since

$$0 \le \frac{1}{\sqrt{n^3 + 30n}} \le \frac{1}{\sqrt{n^3}} = \frac{1}{n^{\frac{3}{2}}}$$

and since $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges the original series converges also and hence absolutely converges.

(b) $\sum_{n=2}^{\infty} (-1)^n \frac{n+3}{n}$

Solution: For large values of n, $\frac{n+3}{n} \sim \frac{1}{1} = 1$ and so we do not expect this series to converge. More specifically,

$$\lim_{n\to\infty} \frac{n+3}{n} = \lim_{n\to\infty} \frac{1 + \frac{3}{n}}{1} = 1$$

is not 0 and so by the n-th term test the series diverges.

7. (*11 points each*)

Determine if the series is **absolutely convergent**, **conditionally convergent** or **divergent**. Be sure to show your reasoning. No work, no credit.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{7^n}$

Solution: Since all the terms of this series are positive, the series an only absolutely converge or diverge. We will use the Ratio test:

$$\lim_{n\to\infty} \frac{\frac{(n+1)^2}{7^{n+1}}}{\frac{n^2}{7^n}} = \lim_{n\to\infty} \frac{1}{7}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{7}$$

Since $\frac{1}{7} < 1$ the series absolutely converges by the Ratio test.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n^n)}$

Solution: We will use the integral test. If $f(x) = \ln(x^x) = x \ln x$ then $f'(x) = \ln x + 1 > 0$ for $x \geq 2$ and so $\frac{1}{\ln(x^x)}$ is positive and decreasing (so we can apply the test). To evaluate the integral, we will use a u-substitution with $u = \ln x$:

$$\int \frac{dx}{\ln(x^x)} = \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$$

and so

$$\int_2^{\infty} \frac{d}{\ln(x^x)} = \Big[\lim_{b\to\infty} \ln|\ln x|\Big]_2^b = \lim_{b\to\infty} \ln|\ln b| - \ln(\ln 2)$$

Since $\ln x \to \infty$ as $x \to \infty$, $\lim_{b\to\infty} \ln|\ln b| = \infty$ and the integral is divergent. Thus, by the Integral Test, the series diverges also.

8. (*5 points*) Show that for any number $r \neq 1$ and positive integer k,

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

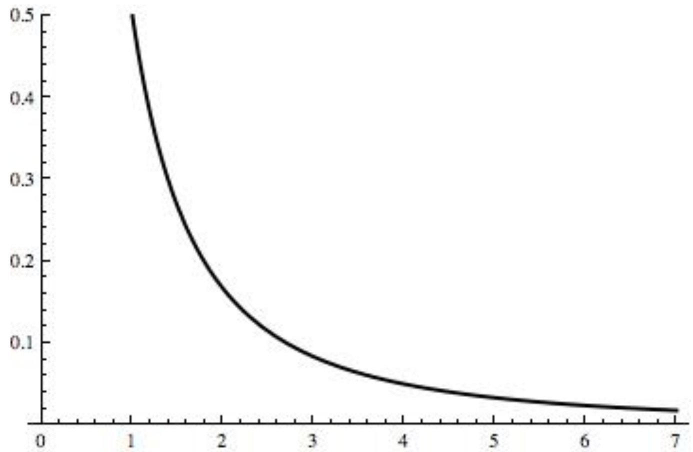
Solution:

$$\begin{aligned}
(1 + r + r^2 + \cdots + r^k)(1 - r) &= 1 \cdot (1 - r) + r(1 - r) + r^2(1 - r) + \cdots + r^k(1 - r) \\
&= (1 - r) + (r - r^2) + (r^2 - r^3) + \cdots + (r^k - r^{k+1}) \\
&= 1 + (-r + r) + (-r^2 + r^2) + \cdots + (-r^k + r^k) - r^{k+1} \\
&= 1 + 0 + 0 + \cdots + 0 - r^{k+1} \\
&= 1 - r^{k+1}
\end{aligned}$$

and if $r \neq 1$ we can divide this equality by $(1 - r)$.

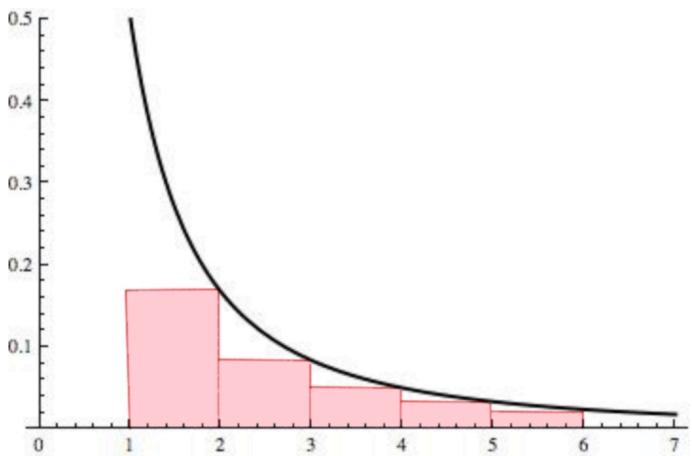
9. (*3 points*) Draw on the diagram and give a brief explanation why

$$\sum_{n=2}^{6} \frac{1}{n(n+1)} \leq \int_1^6 \frac{dx}{x(x+1)}$$



$$f(x) = \frac{1}{x(x+1)}$$

Solution:

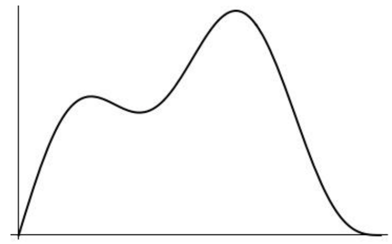


Each of the rectangles has width 1, and their heights are determined by the function $f(x) = \frac{1}{x(x+1)}$ at the values 2, 3, ..., 6 and so the sum of the area of the rectangles is

$$1 \cdot \frac{1}{2(2+1)} + 1 \cdot \frac{1}{3(3+1)} + \cdots + 1 \cdot \frac{1}{6(6+1)} = \sum_{n=2}^{6} \frac{1}{n(n+1)}$$

which by the diagram is less than the total area under the graph from 1 to 6, which is the integral from 1 to 6.

10. (*8 points, 2/3/3*)



This problem concerns the curve

$$y = 2\sin x + \sin 2x, \qquad 0 \le x \le \pi$$

(a) Give an integral for the length of the curve. You do not need to evaluate the integral.

(b) Give an integral for the area of the surface obtained by rotating the curve about the x-axis. You do not need to evaluate the integral.

(c) Give an integral for the area of the surface obtained by rotating the curve about the y-axis. You do not need to evaluate the integral.

Solution: The key thing is to determine that we want to integrate with respect to x and then by factoring out a dx:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

compute

$$\frac{dy}{dx} = 2\cos x + 2\cos 2x$$

(a) $\displaystyle\int ds = \int_0^\pi \sqrt{1 + (2\cos x + 2\cos 2x)^2}\, dx$

(b) $\displaystyle\int y\, ds = \int_0^\pi (2\sin x + \sin 2x)\sqrt{1 + (2\cos x + 2\cos 2x)^2}\, dx$

(c) $\displaystyle\int x\, ds = \int_0^\pi x\sqrt{1 + (2\cos x + 2\cos 2x)^2}\, dx$

11. (*8 points*) Short answer.

(a) Suppose that $c(x) = \sum_{n=0}^{\infty} c_n x^n$ converges for $x = -4$ but diverges for $x = 6$.

i. $\sum_{n=0}^{\infty} (-1)^n c_n$ (absolutely converges/ conditionally converges/diverges).

Solution: Since the series is centered at 0 and converges at $x = -4$ we know that the series coverges absolutely at least for all $x \in (-4, 4)$ and in particular when $x = -1$ and so $\sum_{n=0}^{\infty} (-1)^n c_n$ converges absolutely.

ii. $\sum_{n=0}^{\infty}(-7)^n c_n$ (absolutely converges/ conditionally converges/diverges).

Solution: Since the series is centered at 0 and diverges at $x = 6$ we know that the series diverges at least for all $|x| > 6$ and in particular when $x = -7$ and so $\sum_{n=0}^{\infty}(-7)^n c_n$ diverges.

(b) Calculate the binomial coefficient $\binom{-3}{4} =$

Solution:

$$\binom{-3}{4} = \frac{(-3)(-3-1)(-3-2)(-3-3)}{4!} = \frac{3 \cdot 4 \cdot 5 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 15$$

(c) Recall that $\frac{\pi}{4} = \tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. By the Alternating Series Estimation, how accurate is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = \frac{263}{315}$ to the actual value of $\frac{\pi}{4}$?

Solution: By the Alternating Series estimation, we consider the next term of the series, or $\frac{(-1)^5}{2(5)+1} = -\frac{1}{11}$, and so the sum is an over estimation by no more than $\frac{1}{11}$.

12. (*10 points*) Find a series solution to the integral

$$\int_{-1}^{1} \sin(x^2)\, dx$$

Solution: We recall that

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and that it has infinite radius. Thus, using term-by-term integration

$$\begin{aligned}
\int_{-1}^{1} \sin(x^2)\,dx &= \int_{1}^{1} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \right) dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \int_{-1}^{1} x^{4n+2}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\left. \frac{x^{4n+3}}{4n+3} \right|_{-1}^{1} \right) \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{2}{(4n+3)}
\end{aligned}$$

13. (*10 points each*) Find the radius and interval of convergence for the power series. Be sure to indicate which points converge absolutely and which converge conditionally.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$

Solution: We find the radius by using the Ratio Test:

$$\lim_{n\to\infty} \frac{\left| \frac{(x-2)^{n+1}}{1\cdot3\cdots(2n+1)\cdot(2(n+1)+1)} \right|}{\left| \frac{(x-2)^n}{1\cdot3\cdots(2n+1)} \right|} = |x-2| \lim_{n\to\infty} \frac{1\cdot3\cdots(2n+1)}{1\cdot3\cdots(2n+1)(2n+3)} = |x-2| \lim_{n\to\infty} \frac{1}{2n+3} = 0$$

Thus, the series converges absolutely for all values of x. The radius is ∞ and the interval of convergence is $(-\infty, \infty)$ (there are no endpoints to check).

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n}$

Solution: We find the radius by using the Ratio Test:

$$\lim_{n\to\infty} \frac{\left| \frac{(3x-2)^{n+1}}{n+1} \right|}{\left| \frac{(3x-2)^n}{n} \right|} = |3x-2| \lim_{n\to\infty} \frac{n}{n+1} = |3x-2| \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = |3x-2|$$

The series converges absolutely when $|3x-2|<1$ or $\left|x-\frac{2}{3}\right|<\frac{1}{3}$. Thus the series has radius $\frac{1}{3}$ with center $\frac{2}{3}$. The endpoints are at $\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$ and $\frac{2}{3}+\frac{1}{3}=1$.
When $x=1$ we have the series

$$\sum_{n=0}^{\infty}\frac{(3(1)-2)^n}{n}=\sum_{n=0}^{\infty}\frac{1}{n}$$

which diverges since it is the Harmonic Series (or a p-series with $p=1$).
When $x=\frac{1}{3}$ we have the series

$$\sum_{n=0}^{\infty}\frac{(3(1/3)-2)^n}{n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$$

This is the alternating Harmonic series, which converges conditionally (it converges by the Alternating Series test but does not absolutely converge since the absolute values produce the Harmonic series).
The interval of convergence is $[\frac{1}{3},1)$.

14. (*10 points*) Give the Taylor polynomial with degree 2 centered at 1 for $f(x)=\sqrt[3]{x}$.

Then use Taylor's Inequality to estimate the accuracy of this approximation for $\sqrt[3]{2}$.

Solution: We first compute the first three derivatives of f:

Derivative	Function	At 1
f	$x^{\frac{1}{3}}$	1
f'	$\frac{1}{3}x^{-\frac{2}{3}}$	$\frac{1}{3}$
f''	$-\frac{2}{9}x^{-\frac{5}{3}}$	$-\frac{2}{9}$
$f^{(3)}$	$\frac{10}{27}x^{-\frac{8}{3}}$	

Thus, the degree 2 Taylor polynomial for f at 1 is

$$T_2f=1+\frac{1}{3}(x-1)+\frac{-\frac{2}{9}}{2!}(x-1)^2=1+\frac{(x-1)}{3}-\frac{(x-1)^2}{9}$$

By Taylor's Estimation Theorem we know that

$$f(x) - T_2f(x) = \frac{f^{(3)}(c)}{3!}(x-1)^3$$

for some c between 1 and x. Thus, when $x = 2$

$$\left|\sqrt[3]{2} - T_2f(2)\right| = \left|\frac{\frac{10}{27}c^{-\frac{8}{3}}}{3!}(2-1)^3\right| = \frac{5}{81}\frac{1}{c^{\frac{8}{3}}}$$

for some c between 1 and 2. Since $\frac{8}{3} > 1$, $x^{\frac{8}{3}}$ is increasing for $x > 0$ and then $\frac{1}{x^{\frac{8}{3}}}$ is decreasing on $[1, 2]$. Thus, the largest $\frac{1}{x^{\frac{8}{3}}}$ can be on $[1, 2]$ is when $x = 1$ or $\frac{1}{1^{\frac{8}{3}}} = 1$. So

$$\left|\sqrt[3]{2} - T_2f(2)\right| = \frac{5}{81}\frac{1}{c^{\frac{8}{3}}} \leq \frac{5}{81}$$

15. (*12 points*) Determine a power series centered at 0 for

$$f(x) = \sin^{-1}(x^2)$$

and use it to determine the 104-th derivative of $\sin^{-1} x$ at 0. You may find the following useful

$$\binom{-\frac{1}{2}}{n} = (-1)^n \frac{(1)(3)(5)\cdots(2n-1)}{2^n n!}$$

Solution: We begin by recalling that

$$\frac{d}{dx}\sin^{-1} x^2 = \frac{2x}{\sqrt{1-x^2}} = 2x(1-x^2)^{-\frac{1}{2}}$$

and the Binomial Series

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

which has radius 1. Thus,

$$\begin{aligned}
\sin^{-1} x^2 &= C + \int 2x(1-x^2)^{-1}\,dx \\
&= C + \int \left(\sum_{n=0}^{\infty} 2x \binom{-\frac{1}{2}}{n} (-x^2)^n \right) dx \\
&= C + \sum_{n=0}^{\infty} 2 \binom{-\frac{1}{2}}{n} (-1)^n \int x^{2n+1}\,dx \\
&= C + \sum_{n=0}^{\infty} 2 \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+2}}{2n+2}
\end{aligned}$$

When $x = 0$ we find $0 = \sin^{-1} 0 = C + 0$ and so $C = 0$ and

$$\sin^{-1} x^2 = \sum_{n=0}^{\infty} 2 \binom{-\frac{1}{2}}{n} (-1)^n \frac{x^{2n+2}}{2n+2}$$

for $|x| < 1$. Recall that if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ with radius not 0 then $c_n = \frac{f^{(n)}(a)}{n!}$ or $f^{(n)}(a) = n!c_n$. Thus, the 104-th derivative is 104! times the coefficient in our power series corresponding to x to the 104-th power, or $2n+2 = 104$ or $n = 101$. Thus, the 104-th derivative is

$$104! \cdot 2 \binom{-\frac{1}{2}}{101} (-1)^{101} \frac{1}{2(101)+2} = 104! \cdot 2 \cdot (-1)^{101} \frac{(1)(3)(5)\cdots(2(100)-1)}{2^{100}100!} (-1)^{101} \frac{1}{2(101)+2}$$
$$= \frac{(101)(103)(104)(1)(3)(5)\cdots(199)}{2^{100}}$$

16. (*9 points, 3 points each*) Recall that the follow parametric equations model a particle traveling counter-clockwise about an ellipse at one rev$/2\pi$ unit time starting at the point $(a, 0)$.

$$x = a\cos t \qquad y = b\sin t$$

(a) What is the slope of the tangent line when $t = \frac{\pi}{4}$?

Solution: The slope of the tangent line is given by $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{b\cos t}{-a\sin t}$$

When $t = \frac{\pi}{4}$ the slope is $\frac{b\cos(\frac{\pi}{4})}{-a\sin(\frac{\pi}{4})} = -\frac{b}{a}$.

(b) Give an integral for the area of the ellipse using this set of parametric equations. You do not need to solve the integral.

Solution: The two issues here are the bounds of integration and the orientation. Considering when $t = 0, \frac{\pi}{2}, \pi$ we see that as t goes from 0 to π we trace out along the top edge of the ellipse. However, $\frac{dx}{dt} = -a\sin t \leq 0$ and so the orientation is reversed (use $-dt$). Hence:

$$A = 2\int y\,dx = 2\int_0^{\pi} b\sin t(-a\sin t)(-dt)$$

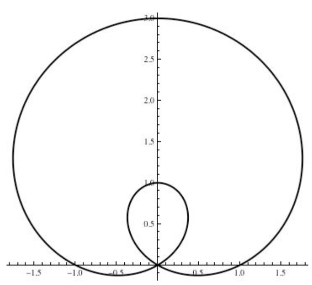
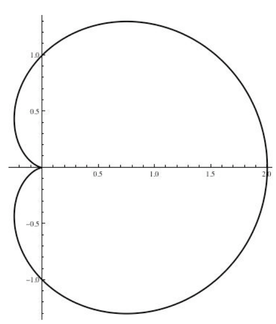
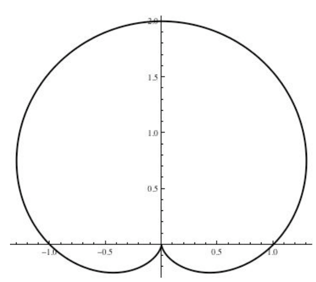
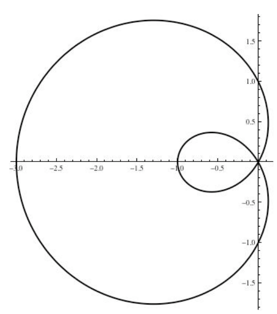
(c) Give an integral for the circumference of the ellipse using this set of parametric equations. You do not need to solve the integral.

Solution: As t goes from 0 to 2π we trace once about the ellipse. Thus,

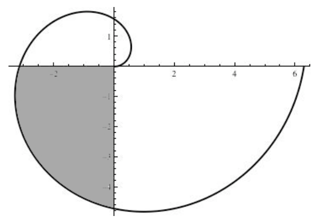
$$L = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (b\cos t)^2}\,dt$$

17. (*4 points*) Match the graphs to their corresponding polar equations. (Mistake on the Exam, only two of these match–correction was made during the exam).

$$r = \sin\theta \quad r = \cos\theta \quad r = 1 + 2\sin\theta \quad r = 1 - 2\cos\theta$$

18. (*5 points*) Find the area of the shaded region of $r = e^{\theta}$.



Solution: To find area in polar coordinates

$$A = \int \frac{1}{2} r^2 \, d\theta$$

The bounds of integration are from π to $\frac{3\pi}{2}$ and so

$$\begin{aligned} A &= \int_{\pi}^{\frac{3\pi}{2}} \frac{1}{2} \left(e^{\theta}\right)^2 \, d\theta \\ &= \frac{1}{2} \int_{\pi}^{\frac{3\pi}{2}} e^{2\theta} \, d\theta \\ &= \frac{1}{4} e^{2\theta} \Big|_{\pi}^{\frac{3\pi}{2}} = \frac{1}{4} \left(e^{3\pi} - e^{2\pi}\right) \end{aligned}$$

Extra Credit (*10 points*)

Recall that hyperbolic cosine is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

If we let the imaginary number $i = \sqrt{-1}$, use power series to establish the identity

$$\cos(ix) = \cosh(x)$$

Solution: Recall that

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and so

$$\cos(ix) = \sum_{n=0}^{\infty}(-1)^n \frac{(ix)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n (i)^{2n} \frac{x^{2n}}{(2n)!}$$

Now $(i)^{2n} = (i^2)^n = (-1)^n$, and $(-1)^n(-1)^n = (-1)^{2n} = ((-1)^2)^n = 1$. Thus, $(-1)^n(i)^{2n} = 1$ and

$$\cos(ix) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

On the other hand, recalling that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we compute

$$\begin{aligned}
\cosh(x) &= \frac{1}{2}\left(e^x + e^{-x}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0} \frac{x^n + (-1)^n x^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(1 + (-1)^n)x^n}{2 \cdot n!}
\end{aligned}$$

When n is even $(1 + (-1)^n) = 2$ and when n is odd $(1 + (-1)^n) = 0$. Thus, this series has only even terms $n = 2k$ and so

$$\cosh x = \sum_{k=0}^{\infty} \frac{2 \cdot x^{2k}}{2 \cdot (2k)!} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$