Math 241: Exam #2

Name:

NetID:

- When space is provided, **show work which justifies your answer**. You do not need to show work on multiple choice questions unless otherwise specified.
- No calculators, notes, books, etc... are permitted.
- You do not need to numerically evaluate expressions such as $\sqrt{7}, 4/13, \cos(\pi/10)$, etc...
- The exam lasts **60 minutes**, has **6 pages** and consists of **7 questions**.

Question 1.

(9 points) The function f is differentiable and its second derivatives exist and are continuous. The table contains the values of f and its first and second-order partial derivatives at the points A, B, C, and D.

	f	f_x	f_y	f_{xx}	f_{xy}	f_{yy}
A(0,0)	0	0	1	0	2	1
B(1,1)	2	0	0	1	0	2
C(1,2)	1	0	0	1	1	0
D(2,2)	0	-1	0	0	0	0

(a) Which of the points $A(0,0)$, $B(1,1)$, $C(1,2)$, $D(2,2)$ are critical points? Mark all that apply.

☐ A ☒ B ☒ C ☐ D

(b) Use the second derivatives test to determine whether the critical points of f (that you found in part (a)) are local minima, local maxima, or saddle points. Write DNE if the corresponding type of critical point does not occur at the points A through D.
Show your work.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

at B: $D(B) = 1 \cdot 2 - 0^2 = 2 > 0$, and $f_{xx} > 0$,
so this is a local minimum

at C: $D(C) = 1 \cdot 0 - 1 = -1 < 0$, so this is a saddle pt.

f has a local minimum at the point(s) B

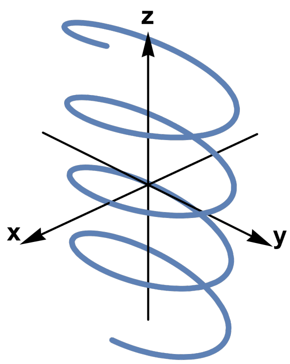
f has a local maximum at the point(s)

f has saddle point(s) at C

(c) Exactly one of the following statements is correct. Which?

- ○ the function f has an absolute maximum in the disk $\{(x,y) \mid x^2 + y^2 < 1\}$
- ● the function f has an absolute maximum in the triangle $\{(x,y) \mid x \geq 0, y \geq 0, x + y \leq 5\}$
- ○ the function f has an absolute maximum in the whole plane $\mathbb{R}^2$

Question 2. **(2 points)** Which of the following is a parametrization of the depicted curve? Mark your answer.



- ○ $\mathbf{r}(t) = \langle \cos t, 2\sin t, 3\cos t \rangle, \quad 0 \le t \le 2\pi$
- ○ $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, \cos(3t) \rangle, \quad 0 \le t \le 2\pi$
- ● $\mathbf{r}(t) = \langle \cos(4t), 2\sin(4t), t - \pi \rangle, \quad 0 \le t \le 2\pi$
- ○ $\mathbf{r}(t) = \langle t\cos(4t), t, t\sin(4t) \rangle, \quad 0 \le t \le 2\pi$

Question 3. **(4 points)** Consider the vector field $\mathbf{F}(x, y) = \underbrace{(6xy^3 + 9)}_{P}\mathbf{i} + \underbrace{ax^2y^2}_{Q}\mathbf{j}$, where a is some real number. For what value(s) of a is $\mathbf{F}$ conservative?

$\vec{F} = \langle P, Q \rangle$

For $\vec{F}$ to be conservative, we need $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

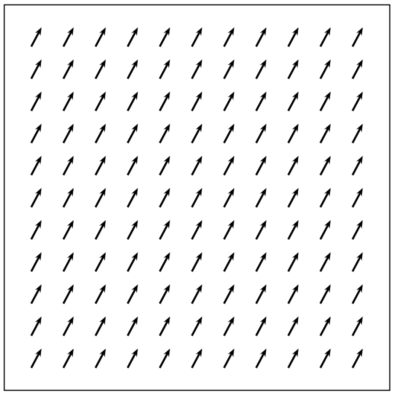
$$\frac{\partial P}{\partial y} = 18xy^2 \overset{?}{=} 2axy^2 = \frac{\partial Q}{\partial x}$$

Equality holds exactly when $a = 9$.

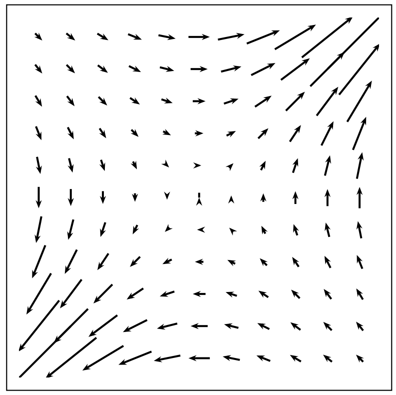
$a =$ $\boxed{9}$

Question 4. **(4 points)** Exactly two of the following vector fields are *not* conservative. Which two?

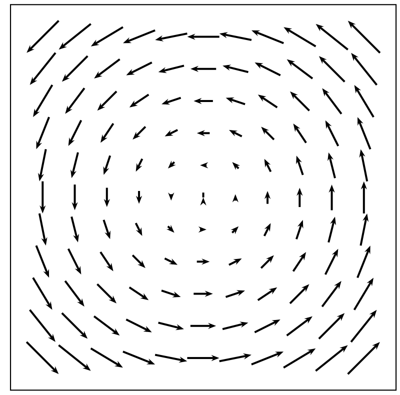
☐ A ☐ B ☒ C ☐ D ☒ E ☐ F



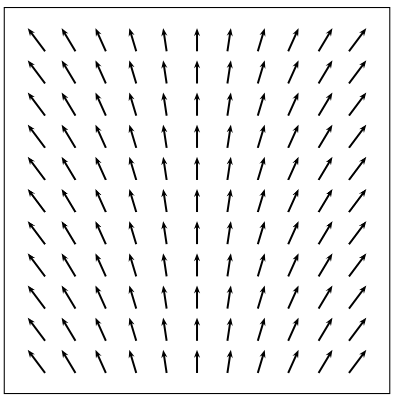
A



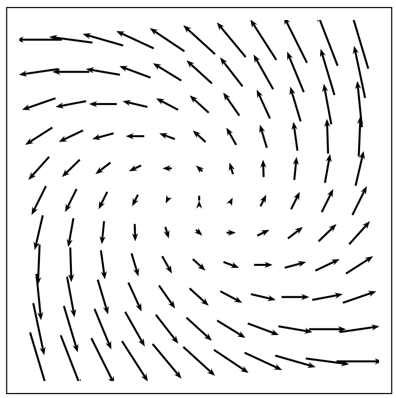
B



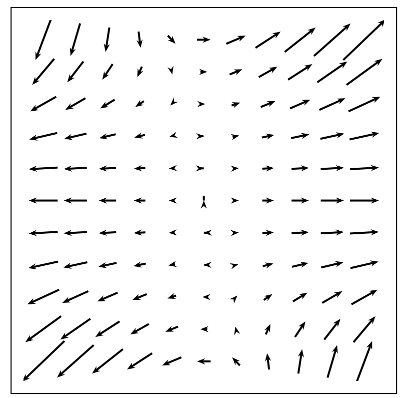
C



D



E



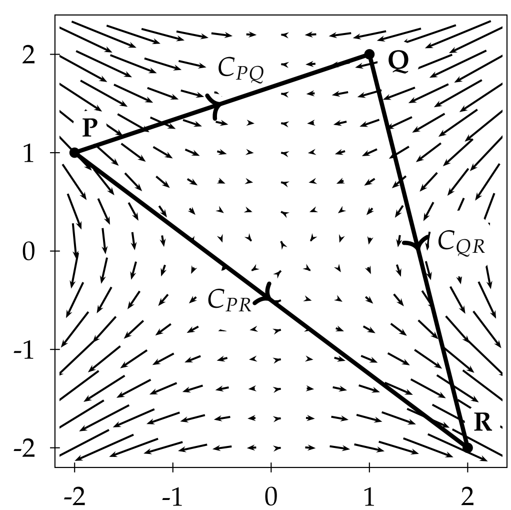
F

Question 5. **(5 points)** Let $g(x, y)$ be differentiable and consider the vector field $\mathbf{G} = \nabla g$. Suppose C_{PQ} is the line segment from $P(-2, 1)$ to $Q(1, 2)$, C_{QR} is the line segment from $Q(1, 2)$ to $R(2, -2)$, and C_{PR} the line segment from $P(-2, 1)$ to $R(2, -2)$. It is known that $\int_{C_{PQ}} \mathbf{G} \cdot d\mathbf{r} = 3$ and $\int_{C_{QR}} \mathbf{G} \cdot d\mathbf{r} = 0$.

Compute $\int_{C_{PR}} \mathbf{G} \cdot d\mathbf{r}$.

By path independence

$$\int_{C_{PR}} \vec{G} \cdot d\vec{r} = \int_{C_{PQ}} \vec{G} \cdot d\vec{r} + \int_{C_{QR}} \vec{G} \cdot d\vec{r} = 3 + 0 = 3$$



$\int_{C_{PR}} \mathbf{G} \cdot d\mathbf{r} =$ $\boxed{3}$

Question 6. **(8 points)** Set up and evaluate an integral to compute the area of a fence built over the parametrized curve $\mathbf{r}(t) = \langle \sin(3t), \cos(3t) \rangle$, with $t \in [0, \pi/6]$, where the height is described by the function $f(x, y) = x^2 y + 2$.

$$\text{Area} = \int_C f \, ds \text{ , where } C \text{ is the parametrized curve } \vec{r}(t) = \langle \sin(3t), \cos(3t) \rangle .$$

$$\vec{r}\,'(t) = \langle 3\cos(3t), -3\sin(3t) \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2(3t) + 9\sin^2 3t} = 3 .$$

$$\rightsquigarrow \text{Area} = \int_0^{\pi/6} \underbrace{(\sin^2 3t \cos 3t + 2)}_{f(x(t),y(t))} \cdot \underbrace{3\,dt}_{ds} = 6\int_0^{\pi/6} dt + \int_0^{\pi/6} \sin^2 3t \, d(\sin 3t) =$$

$$= \pi + \left. \frac{\sin^3 3t}{3} \right|_{t=0}^{\frac{\pi}{6}} = \pi + \frac{1}{3}$$

Area= $\boxed{\pi + 3}$

Question 7. **(8 points)** Use Lagrange multipliers to find the absolute minimum and absolute maximum of the function $f(x,y) = 2x - y + 2$ subject to the constraint $g(x,y) = x^2 + \frac{y^2}{2} = 2$.

Need to solve $\nabla f = \lambda \nabla g$, or $\nabla g = \vec{0}$.

Note: $\nabla g = \langle 2x, y \rangle$ is zero only at $(0,0)$, which doesn't satisfy the constraint.

So we may assume $\nabla g \neq \vec{0}$.

$\nabla f = \langle 2, -1 \rangle = \lambda \langle 2x, y \rangle = \lambda \nabla g$

means $\begin{cases} 2 = 2\lambda x \\ -1 = \lambda y \end{cases}$ so $\begin{cases} \lambda x = 1 \\ \lambda y = -1 \end{cases}$

these equations can't be satisfied if $\lambda = 0$, so we may assume $\lambda \neq 0$.

then $x = \frac{1}{\lambda}$, $y = -\frac{1}{\lambda}$

Plugging into the constraint, we get

$g(x,y) = x^2 + \frac{y^2}{2} = \frac{1}{\lambda^2} + \frac{1}{2\lambda^2} = \frac{3}{2\lambda^2} = 2$

So $\lambda^2 = \frac{3}{4}$, $\lambda = \pm \frac{\sqrt{3}}{2}$

$\longrightarrow$ if $\lambda = \frac{\sqrt{3}}{2}$ $(x,y) = \left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$ and $f\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}} + \frac{2}{\sqrt{3}} + 2 = 2\sqrt{3} + 2$

$\longrightarrow$ if $\lambda = -\frac{\sqrt{3}}{2}$ $(x,y) = \left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$ and $f\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = -\frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}} + 2 = -2\sqrt{3} + 2$

minimum value of f = $\boxed{-2\sqrt{3} + 2}$

at the point(s) $\boxed{\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)}$

maximum value of f = $\boxed{2\sqrt{3} + 2}$

at the point(s) $\boxed{\left(\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)}$