Math 241: Exam #2

Name:

NetID:

- When space is provided, **show work which justifies your answer**. You do not need to show work on multiple choice questions unless otherwise specified.
- No calculators, notes, books, etc... are permitted.
- You do not need to numerically evaluate expressions such as $\sqrt{7}$, $4/13$, $\cos(\pi/10)$, etc...
- The exam lasts **60 minutes**, has **6 pages** and consists of **7 questions**.

Question 1. Consider the function $f(x, y) = 3x^2 - 3xy$. **(9 points)**

(a) Find one critical point P of f.

Need to solve $\nabla f = \langle 6x - 3y, -3x \rangle = \langle 0, 0 \rangle$

$$\begin{cases} -3x = 0 \\ 6x - 3y = 0 \end{cases} \leadsto x = 0,\ y = 0$$

critical point $P =$ $\boxed{(0, 0)}$

(b) Use the Second Derivatives test to determine whether the critical point P is

- ○ a local minimum of f,
- ○ a local maximum of f,
- ● a saddle point of f, or
- ○ none of the above?

Show your work.

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 0 \end{vmatrix} = -9 \quad < 0$$

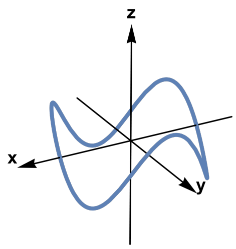
(c) Does the function f have an absolute minimum in the closed unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$?
There is no need to determine the value, if it exists.

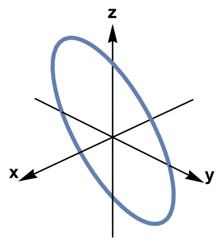
- ● Yes
- ○ No
- ○ It is impossible to tell from the given information.

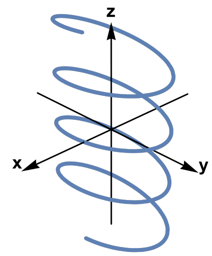
Question 2. Which of the following figures depicts the parameterized curve

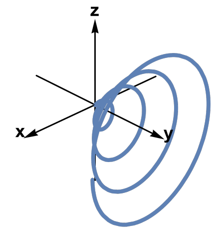
$$\mathbf{r}(t) = \langle \cos t, 2\sin t, 3\cos t\rangle, \qquad 0 \le t \le 2\pi?$$

Mark your answer. **(2 points)**









Question 3. Is the vector field $\mathbf{F}(x, y) = \langle 2xy, x^2 - 2y\rangle$ conservative? Circle your answer: **Yes** **No**

If $\mathbf{F}$ is conservative, find a potential function $f(x, y)$ for $\mathbf{F}$.
If there is no potential function, explain why not and leave the answer box blank. **(5 points)**

Suppose $\vec{F} = \nabla f$, so $\begin{cases} f_x = 2xy \\ f_y = x^2 - 2y \end{cases}$

Then $f = \int 2xy \, dx = x^2 y + g(y)$ for some function $g(y)$.

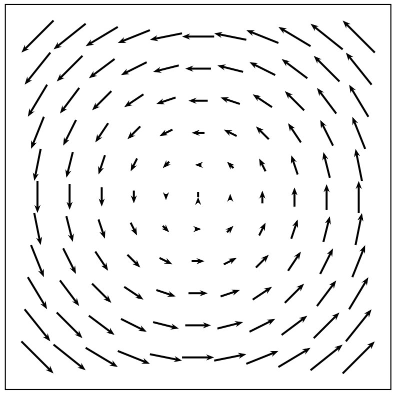
Now $f_y = x^2 + g'(y) = x^2 - 2y$, so $g'(y) = -2y$.

$\rightsquigarrow g(y) = \int -2y \, dy = -y^2 + C$, for some constant C.

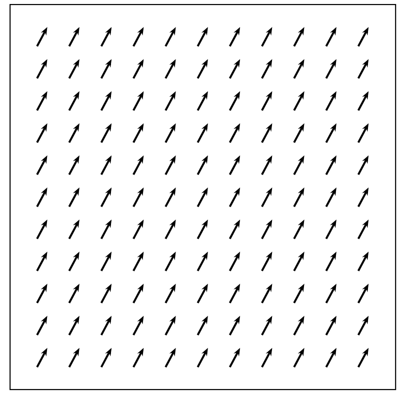
$f(x, y) =$ $\boxed{x^2 y - y^2 + C}$

Question 4. Exactly two of the following vector fields are *not* conservative? Which two? **(4 points)**

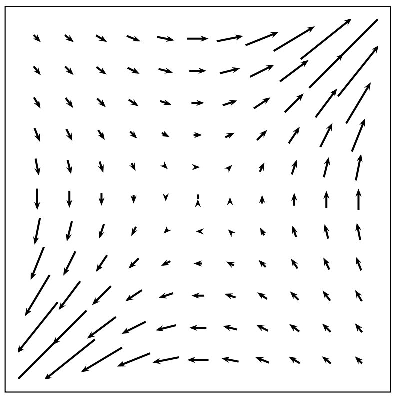
- ☒ A
- ☐ B
- ☐ C
- ☒ D
- ☐ E
- ☐ F



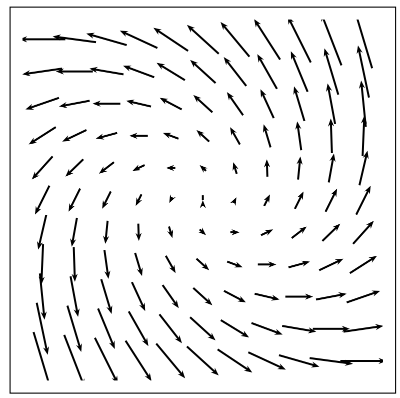
A



B



C



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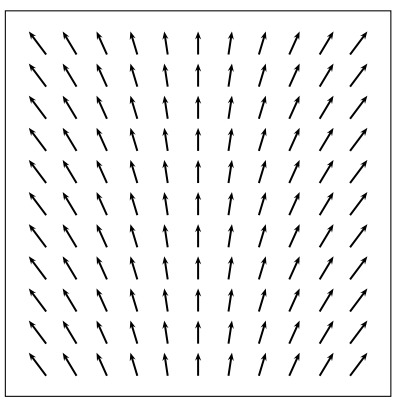
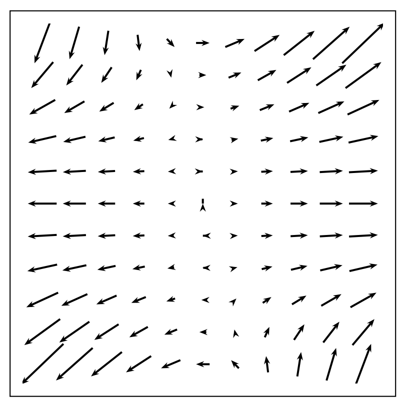


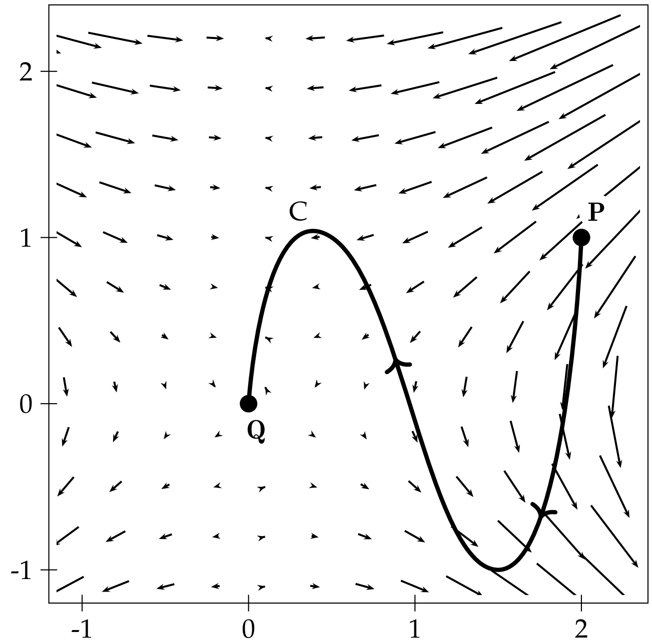
E



F

Question 5. Let $g(x, y) = xy - 3x^2y$. The diagram below depicts the vector field $\mathbf{G} = \nabla g$ and the curve C from the point $P(2, 1)$ to the origin $Q(0, 0)$. **(4 points)**

Compute $\int_C \mathbf{G} \cdot d\mathbf{r}$.



By FTLI,

$$\int_C \vec{G} \cdot d\vec{r} = g(Q) - g(P) =$$

$$= g(0,0) - g(2,1) = 0 - 2 + 12 = 10$$

$\int_C \mathbf{G} \cdot d\mathbf{r} =$ $\boxed{10}$

Question 6. Consider the function $f(x, y) = xy$ and the curve C given by $\mathbf{r}(t) = \langle 2\sin(t), 2\cos(t)\rangle$, $0 \le t \le \pi/2$. Compute $\int_C f(x, y)\, ds$. **(8 points)**

$$\vec{r}'(t) = \langle 2\cos t, -2\sin t \rangle$$

$$|\vec{r}'(t)| = \sqrt{4\cos^2 t + 4\sin^2 t} = 2$$

$$\int_C f\, ds = \int_0^{\pi/2} f(2\sin t, 2\cos t)\, |\vec{r}'(t)|\, dt = \int_0^{\pi/2} 8 \sin t \cos t\, dt = \int_0^{\pi/2} 8 \sin t\, d\sin t =$$

$$= 4\sin^2 t \Big|_{t=0}^{\pi/2} = 4$$

$\int_C f(x, y) ds =$ $\boxed{4}$

Question 7. Use Lagrange multipliers to find the absolute minimum and the absolute maximum of the function

$$f(x,y) = 2x - 4y + 1,$$

subject to the constraint $g(x,y) = x^2 + 2y^2 = 3$. **(8 points)**

Need to solve $\nabla f = \lambda \nabla g$, or $\nabla g = \vec{0}$.

Note: $\nabla g = \langle 2x, 4y \rangle$ is zero only at $(0,0)$, which doesn't satisfy the constraint.

So we may assume $\nabla g \neq \vec{0}$.

$\nabla f = \langle 2, -4 \rangle = \lambda \langle 2x, 4y \rangle = \lambda \nabla g$

means $\begin{cases} 2 = 2\lambda x \\ -4 = 4\lambda y \end{cases}$ so $\begin{cases} \lambda x = 1 \\ \lambda y = -1 \end{cases}$

these equations can't be satisfied if $\lambda = 0$, so we may assume $\lambda \neq 0$.

then $x = \frac{1}{\lambda}$, $y = -\frac{1}{\lambda}$.

Plugging into the constraint, we get

$g(x,y) = x^2 + 2y^2 = \frac{1}{\lambda^2} + \frac{2}{\lambda^2} = \frac{3}{\lambda^2} = 3$

So $\lambda = \pm 1$.

$\longrightarrow$ if $\lambda = 1$, $(x,y) = (1,-1)$ and $f(1,-1) = 2+4+1 = 7$.

$\longrightarrow$ if $\lambda = -1$, $(x,y) = (-1,1)$ and $f(-1,1) = -2-4+1 = -5$

minimum value of f = -5

at the point(s) $(-1,1)$

maximum value of f = 7

at the point(s) $(1,-1)$