Math 241: Exam #2

Name:

NetID:

- When space is provided, **show work which justifies your answer**. You do not need to show work on multiple choice questions unless otherwise specified.
- No calculators, notes, books, etc... are permitted.
- You do not need to numerically evaluate expressions such as $\sqrt{7}$, $4/13$, $\cos(\pi/10)$, etc...
- The exam lasts **60 minutes**, has **6 pages** and consists of **7 questions**.

Question 1. Consider the function $f(x, y) = 2xy + y^2$. **(9 points)**

(a) Find one critical point P of f.

Need to solve $\nabla f = \langle 2y, 2x+2y \rangle = \langle 0,0 \rangle$

$$\begin{cases} 2y = 0 \\ 2x + 2y = 0 \end{cases} \rightsquigarrow x=0,\ y=0$$

critical point P = $\boxed{(0,0)}$

(b) Use the Second Derivatives test to determine whether the critical point P is

- ○ a local minimum of f,
- ○ a local maximum of f,
- ● a saddle point of f, or
- ○ none of the above?

Show your work.

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = -4 < 0$$

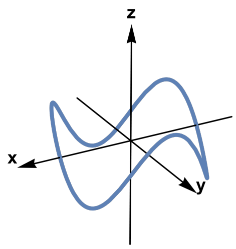
(c) Does the function f have an absolute minimum in the closed unit disk $\{(x, y) \mid x^2 + y^2 \leq 1\}$?
There is no need to determine the value, if it exists.

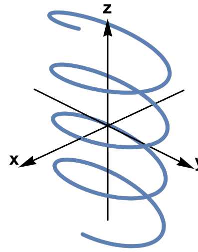
- ● Yes
- ○ No
- ○ It is impossible to tell from the given information.

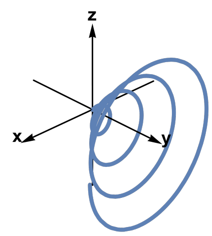
Question 2. Which of the following figures depicts the parameterized curve

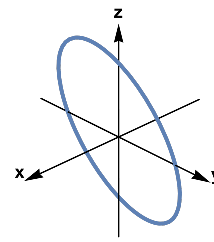
$$\mathbf{r}(t) = \langle \cos t, 2\sin t, 3\cos t \rangle, \qquad 0 \le t \le 2\pi?$$

Mark your answer. **(2 points)**









Question 3. Is the vector field $\mathbf{F}(x, y) = \left\langle 4xy, 2x^2 + 3y^2 \right\rangle$ conservative? Circle your answer: **Yes** **No**

If $\mathbf{F}$ is conservative, find a potential function $f(x, y)$ for $\mathbf{F}$.
If there is no potential function, explain why not and leave the answer box blank. **(5 points)**

Suppose $\vec{F} = \nabla f$, so $\begin{cases} f_x = 4xy \\ f_y = 2x^2 + 3y^2 \end{cases}$

Then $f = \int 4xy \, dx = 2x^2y + g(y)$ for some function $g(y)$.

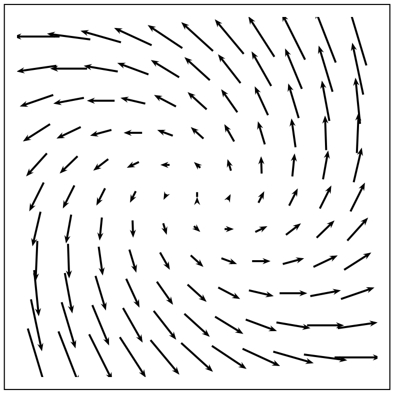
Now $f_y = 2x^2 + g'(y) = 2x^2 + 3y^2$, so $g'(y) = 3y^2$.

$\rightsquigarrow g(y) = \int 3y^2 \, dy = y^3 + C$, for some constant C.

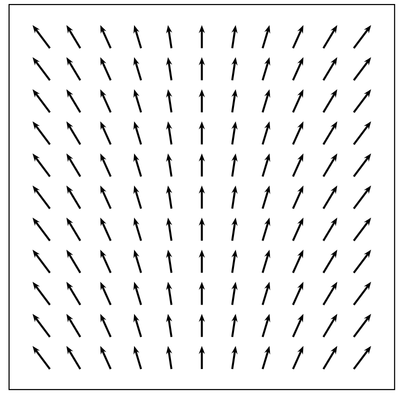
$f(x, y) =$ $\boxed{2x^2y + y^3 + C}$

Question 4. Exactly two of the following vector fields are *not* conservative? Which two? **(4 points)**

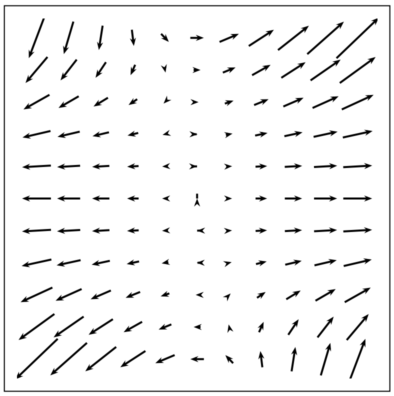
- [x] A
- [ ] B
- [ ] C
- [x] D
- [ ] E
- [ ] F



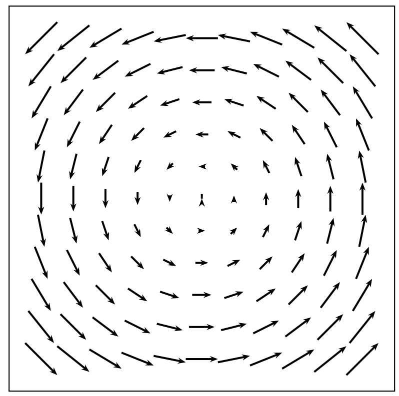
A



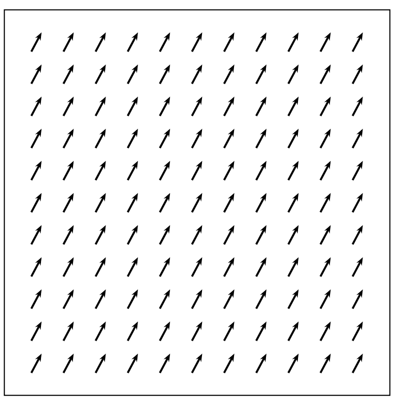
B



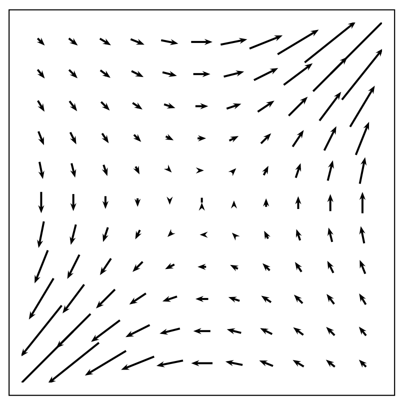
C



D



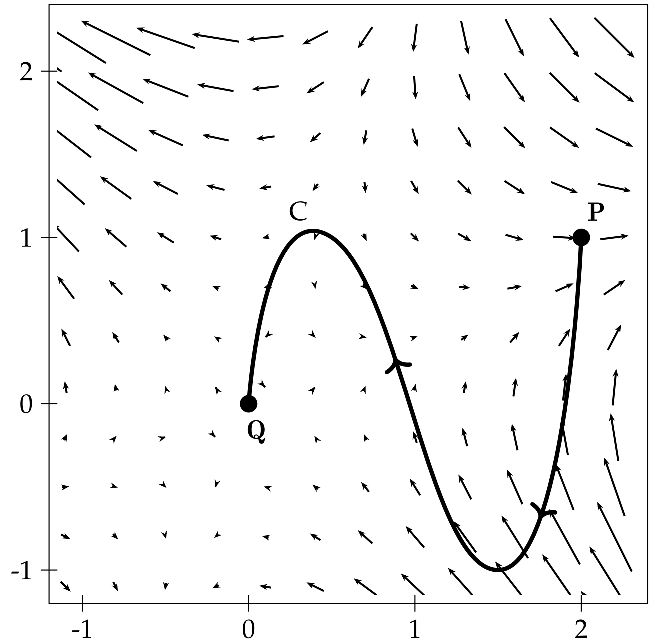
E



F

Question 5. Let $g(x,y) = x^2y - xy^2$. The diagram below depicts the vector field $\mathbf{G} = \nabla g$ and the curve C from the point $P(2,1)$ to the origin $Q(0,0)$. **(4 points)**

Compute $\int_C \mathbf{G} \cdot d\mathbf{r}$.



By FTLI,

$$\int_C \vec{G} \cdot d\vec{r} = g(Q) - g(P) =$$

$$= g(0,0) - g(2,1) = 0 - 2 = -2$$

$\int_C \mathbf{G} \cdot d\mathbf{r} =$ $\boxed{-2}$

Question 6. Consider the function $f(x,y) = xy$ and the curve C given by $\mathbf{r}(t) = \langle -3\sin(t), 3\cos(t) \rangle$, $0 \le t \le \pi/2$. Compute $\int_C f(x,y)\,ds$. **(8 points)**

$$\vec{r}\,'(t) = \langle -3\cos t, -3\sin t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t} = 3$$

$$\int_C f\,ds = \int_0^{\pi/2} f(-3\sin t, 3\cos t)\,|\vec{r}\,'(t)|\,dt = -\int_0^{\pi/2} 27\sin t\cos t\,dt = -\int_0^{\pi/2} 27\sin t\,d\sin t =$$

$$= -\frac{27}{2}\sin^2 t \Big|_{t=0}^{\pi/2} = -\frac{27}{2}$$

$\int_C f(x,y)ds =$ $\boxed{-\frac{27}{2}}$

Question 7. Use Lagrange multipliers to find the absolute minimum and the absolute maximum of the function

$$f(x, y) = -4x + 2y + 5,$$

subject to the constraint $g(x, y) = 2x^2 + y^2 = 3$. **(8 points)**

Need to solve $\nabla f = \lambda \nabla g$, or $\nabla g = \vec{0}$.

Note: $\nabla g = \langle 4x, 2y \rangle$ is zero only at $(0,0)$, which doesn't satisfy the constraint.

So we may assume $\nabla g \neq \vec{0}$.

$\nabla f = \langle -4, 2 \rangle = \lambda \langle 4x, 2y \rangle = \lambda \nabla g$

means $\begin{cases} -4 = 4\lambda x \\ 2 = 2\lambda y \end{cases}$ so $\begin{cases} \lambda x = -1 \\ \lambda y = 1 \end{cases}$

these equations can't be satisfied if $\lambda = 0$, so we may assume $\lambda \neq 0$.

then $x = -\frac{1}{\lambda}$, $y = \frac{1}{\lambda}$

Plugging into the constraint, we get

$g(x, y) = 2x^2 + y^2 = 2 \cdot \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = \frac{3}{\lambda^2} = 3$

So $\lambda = \pm 1$.

$\longrightarrow$ if $\lambda = 1$ $(x, y) = (-1, 1)$ and $f(-1, 1) = 4 + 2 + 5 = 11$

$\longrightarrow$ if $\lambda = -1$ $(x, y) = (1, -1)$ and $f(1, -1) = -4 - 2 + 5 = -1$

minimum value of f =	-1
at the point(s)	$(1, -1)$
maximum value of f =	11
at the point(s)	$(-1, 1)$