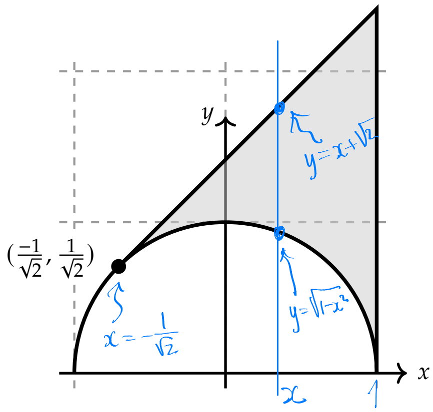
Question 1. Let R be the depicted region above the unit circle $x^2 + y^2 = 1$, below the line $y - x = \sqrt{2}$, and to the left of the line $x = 1$.



(a) **(4 points)** Find the bounds of integration for $\iint_R 2y\, dA$ as an iterated integral

$$\iint_R 2y\, dA = \int_{\boxed{-1/\sqrt{2}}}^{\boxed{1}} \int_{\boxed{\sqrt{1-x^2}}}^{\boxed{x+\sqrt{2}}} 2y\, dy\, dx.$$

Note: The order of integration is already determined.

(b) **(2 points)** Evaluate the integral $\iint_R 2y\, dA$ as set up in part (a).

$$\iint_R 2y\, dA = \int_{-1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x+\sqrt{2}} 2y\, dy\, dx = \int_{-1/\sqrt{2}}^{1} y^2 \Big|_{y=\sqrt{1-x^2}}^{x+\sqrt{2}} dx = \int_{-1/\sqrt{2}}^{1} (x+\sqrt{2})^2 - (1-x^2)\, dx =$$

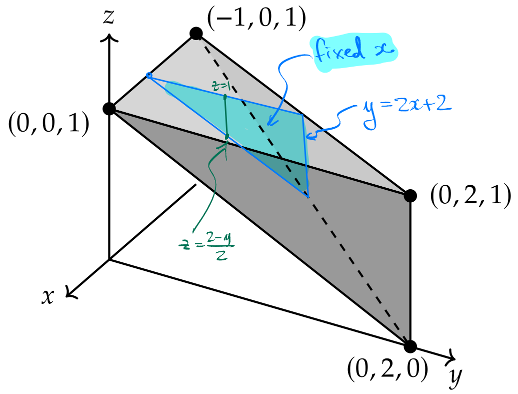
$$= \int_{-1/\sqrt{2}}^{1} 2x^2 + 2\sqrt{2}x + 1\, dx = \frac{2x^3}{3} + \sqrt{2}\, x^2 + x \Big|_{-1/\sqrt{2}}^{1} =$$

$$= \frac{2}{3} + \sqrt{2} + 1 + \frac{2}{3}\frac{1}{2\sqrt{2}} - \sqrt{2}\frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{5}{3} + \frac{7\sqrt{2}}{6}$$

$$\iint_R 2y\, dA = \boxed{\frac{5}{3} + \frac{7\sqrt{2}}{6}}$$

Scratch Space

Question 2. **(6 points)**



Consider the triple integral $\int_0^1 \int_{2-2z}^2 \int_{(y-2)/2}^0 f(x,y,z)\,dx\,dy\,dz$. Its region of integration is depicted; it is bounded by the yz-plane and the planes with equations $z=1$, $2x-y+2=0$, and $y+2z-2=0$.

Determine the limits of integration when changing the order of integration as

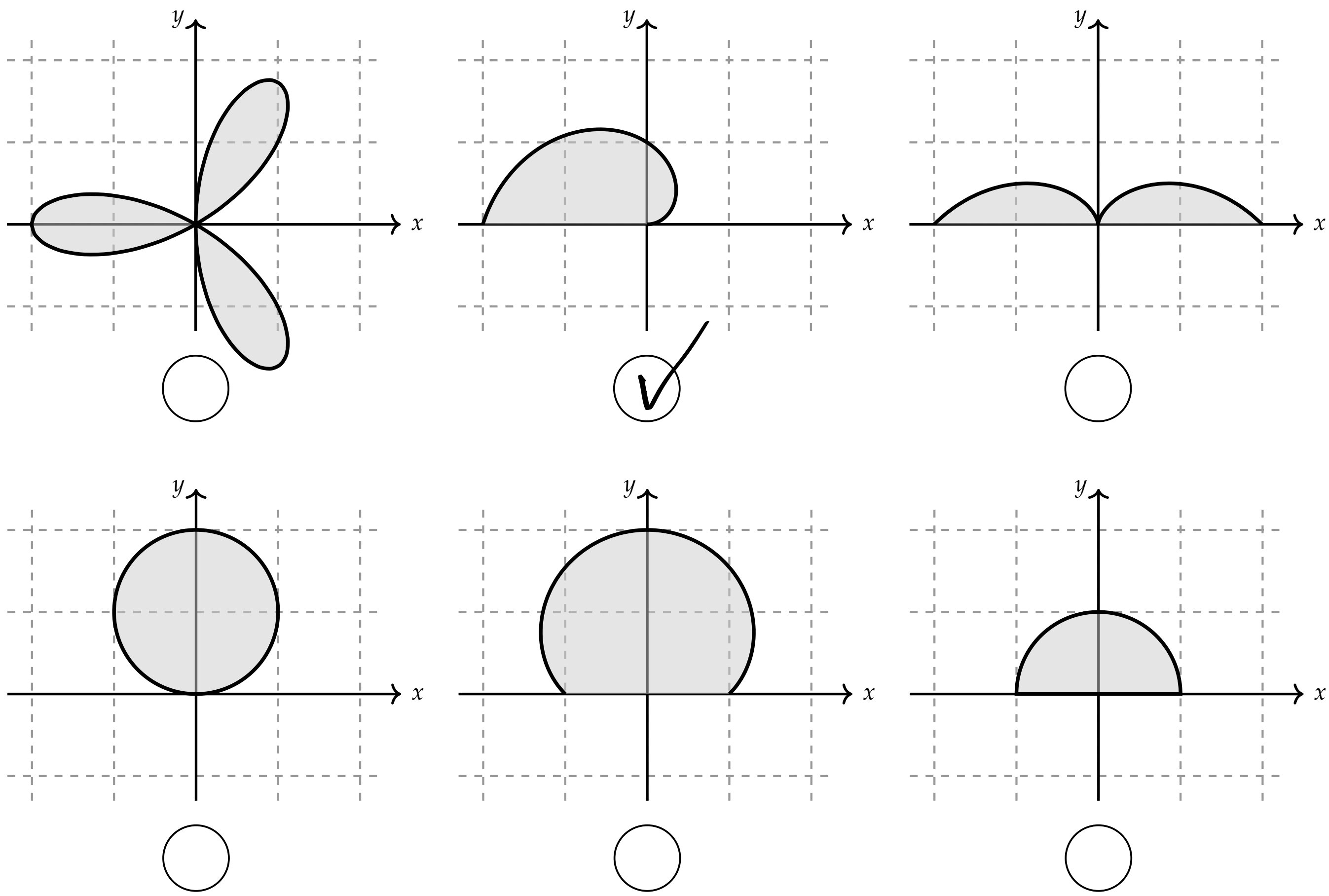
$$\int_0^1 \int_{2-2z}^2 \int_{(y-2)/2}^0 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{\boxed{-1}}^{\boxed{0}} \int_{\boxed{0}}^{\boxed{2x+2}} \int_{\boxed{1-y/2}}^{\boxed{1}} f(x,y,z)\,dz\,dy\,dx.$$

Scratch Space

Question 3. The double integral $\iint_R x - y\, dA$ has the form $\int_0^{\pi} \int_0^{2\theta/\pi} ??\, dr\, d\theta$ when converted into polar coordinates.

(a) **(2 points)** Mark the box of the picture below which depicts the region R in the xy-plane.



(b) **(3 points)** Fill in the missing integrand to convert this integral to polar coordinates. Do **not** compute the integral!

$$\iint_R x - y\, dA = \int_0^{\pi} \int_0^{2\theta/\pi} \boxed{r^2(\cos\theta - \sin\theta)}\, dr\, d\theta$$

Scratch Space

$$(x-y)\,dA = (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

Question 4. **(7 points)** Let R be the region in $\mathbb{R}^3$ that is inside the sphere $x^2 + y^2 + z^2 = 4$, outside of the sphere $x^2 + y^2 + z^2 = 1$, and above the cone $z = -\sqrt{x^2 + y^2}$. Convert the triple integral $\iiint_R z \, dV$ into spherical coordinates. Do **not** compute the integral!

$z \, dV = \rho \cos\phi \cdot \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$

$$\iiint_R z \, dV = \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{1}}^{\boxed{2}} \int_{\boxed{0}}^{\boxed{\frac{3\pi}{4}}} \boxed{\rho^3 \sin\phi \cos\phi} \, d\phi \, d\rho \, d\theta.$$

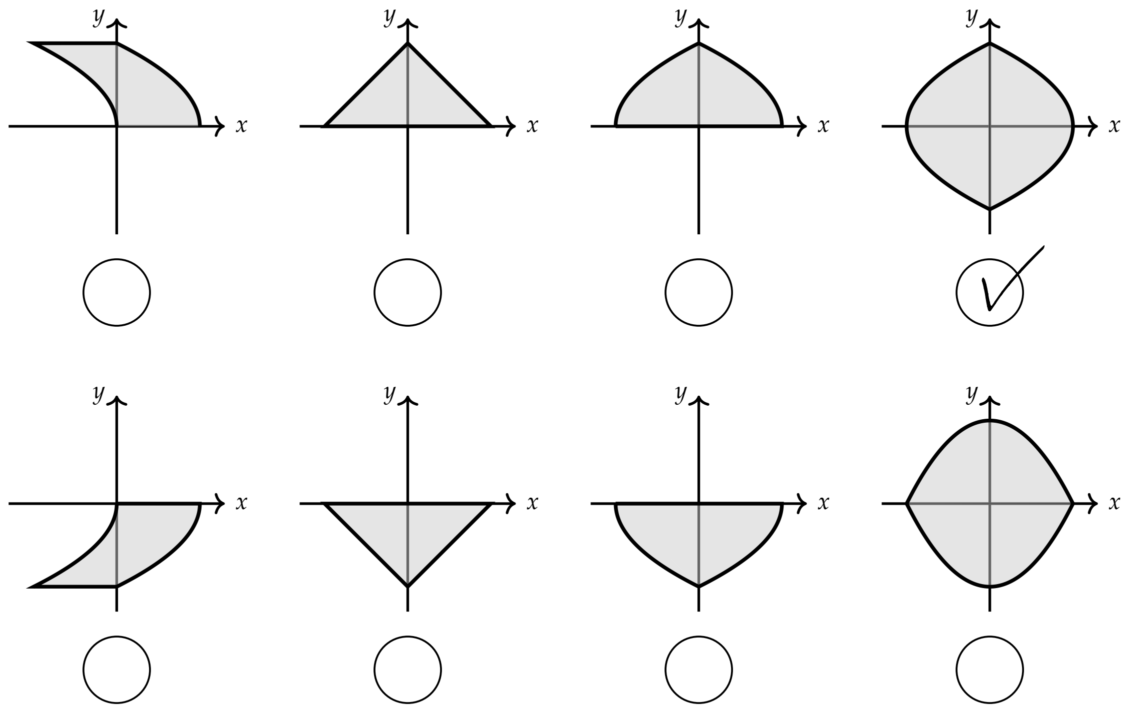
Note: The order of integration is already determined.

Scratch Space

Question 5. Let D be the rectangle $\{(u,v) \mid -1 \le u \le 1 \text{ and } 0 \le v \le 1\}$.
Let $x(u,v) = 2u^2 - 2v^2$ and $y(u,v) = -2uv$. Consider the transformation $T(u,v) = \big(x(u,v), y(u,v)\big) = (2u^2 - 2v^2, -2uv)$.
The transformation T satisfies:

$$T(u,0) = (2u^2, 0)$$
$$T(u,1) = (2u^2 - 2, -2u)$$
$$T(-1,v) = (2 - 2v^2, 2v)$$
$$T(1,v) = (2 - 2v^2, -2v)$$

(a) **(2 points)** Mark the picture which depicts the image $T(D)$.



(b) **(2 points)** Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4u & -2v \\ -4v & -2u \end{vmatrix} = -8u^2 - 8v^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{-8u^2 - 8v^2}$$

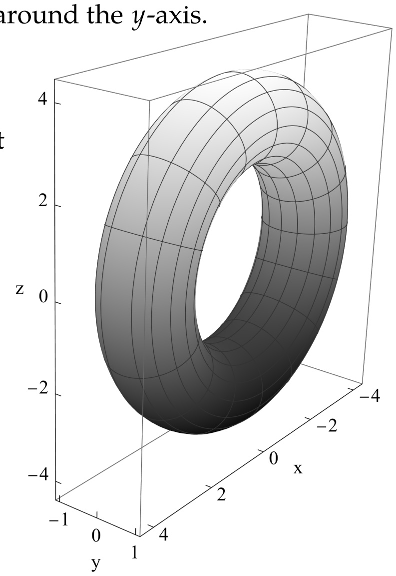
(c) **(3 points)** Set up an iterated integral computing the area of $T(D)$. Do **not** compute the integral!

$$\text{Area}(T(D)) = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{-1}}^{\boxed{1}} \boxed{8u^2 + 8v^2} \, du \, dv$$

Question 6. Let T be the surface parametrized by

$$\mathbf{r}(u,v) = ((3+\cos u)\cos v, \sin u, (3+\cos u)\sin v), \quad \text{for } u, v \text{ in } [0, 2\pi].$$

The surface T can be obtained by revolving the circle $(x-3)^2 + y^2 = 1$ around the y-axis.



(a) **(4 points)** Find the equation for the tangent plane to T at the point $P = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) = \mathbf{r}(\pi, 3\pi/4)$.

A normal vector is given by $\vec{r}_u(\pi, \frac{3\pi}{4}) \times \vec{r}_v(\pi, \frac{3\pi}{4})$

$\vec{r}_u = \langle -\sin u \cos v, \cos u, -\sin u \sin v \rangle$

$\rightsquigarrow \vec{r}_u(\pi, \frac{3\pi}{4}) = \langle 0, -1, 0 \rangle$

$\vec{r}_v = \langle -(3+\cos u)\sin v, 0, (3+\cos u)\cos v \rangle$

$\rightsquigarrow \vec{r}_v(\pi, \frac{3\pi}{4}) = \langle -\sqrt{2}, 0, -\sqrt{2} \rangle$

$\rightsquigarrow \vec{r}_u(\pi, \frac{3\pi}{4}) \times \vec{r}_v(\pi, \frac{3\pi}{4}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 0 \\ -\sqrt{2} & 0 & -\sqrt{2} \end{vmatrix} = \langle \sqrt{2}, 0, -\sqrt{2} \rangle = \vec{n}$

Plane equation: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\rightsquigarrow \sqrt{2}\langle 1, 0, -1 \rangle \cdot \langle x + \frac{\sqrt{2}}{2}, y, z - \frac{\sqrt{2}}{2} \rangle = 0$ $\quad x + \frac{\sqrt{2}}{2} - z + \frac{\sqrt{2}}{2} = 0$

Tangent plane: $\boxed{x - z = -\sqrt{2}}$

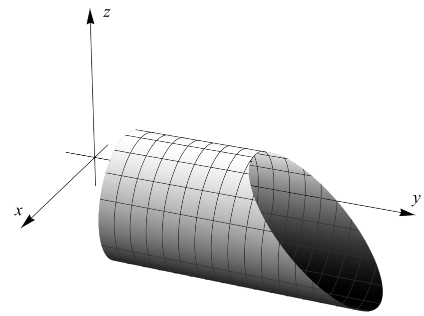
(b) **(2 points)** Which of the following statements is true? You do **not** need to use the parameterization to calculate the integrals.

- [x] $\iint_T x^2\, dS > \iint_T y^2\, dS$
- [ ] $\iint_T x^2\, dS = \iint_T y^2\, dS$
- [ ] $\iint_T x^2\, dS < \iint_T y^2\, dS$

Scratch Space

Question 7. **(3 points)** Let S be the part of the cylinder $(x-1)^2+z^2=1$ between the planes $y=1$ and the plane $y+z=4$. Parameterize S with a function $\mathbf{r}(u,v)$. Be sure to specify the domain D of your parameterization.

$$\left.\begin{aligned} x-1 &= \cos u \\ z &= \sin u \end{aligned}\right\} \quad u = \text{polar angle in } y\text{-}z\text{-plane projection}$$



$1 \le y \le 4-z$

Take $v = y$

$\mathbf{r}(u,v) = \langle 1+\cos u, \quad v, \quad \sin u \rangle$

$D = \{(u,v) \mid 0 \le u \le 2\pi, \quad 1 \le v \le 4-\sin u\}$

Scratch Space