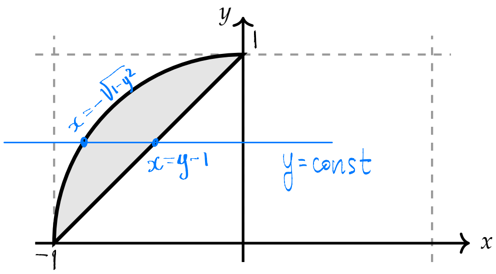
Question 1. Let R be the depicted region above the line $x - y + 1 = 0$ and inside unit circle centered at the origin.



(a) **(4 points)** Find the bounds of integration for $\iint_R 2x\ dA$ as an iterated integral

$$\iint_R 2x\ dA = \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{-\sqrt{1-y^2}}}^{\boxed{y-1}} 2x\, dx\, dy.$$

Note: The order of integration is already determined.

(b) **(2 points)** Evaluate the integral $\iint_R 2x\ dA$ as set up in part (a).

$$\iint_R 2x\, dA = \int_0^1 \int_{-\sqrt{1-y^2}}^{y-1} 2x\, dx\, dy = \int_0^1 x^2 \Big|_{x=-\sqrt{1-y^2}}^{y-1} dy =$$

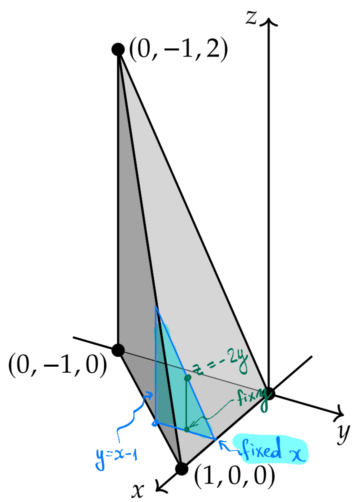
$$= \int_0^1 (y-1)^2 - (1-y^2)\, dy = \int_0^1 2y^2 - 2y\, dy =$$

$$= \frac{2y^3}{3} - y^2 \Big|_0^1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

$$\iint_R 2x\ dA = \boxed{-\frac{1}{3}}$$

Scratch Space

Question 2. **(6 points)**



Consider the triple integral $\int_0^2 \int_{-1}^{-z/2} \int_0^{1+y} f(x, y, z)\, dx\, dy\, dz$. Its region of integration is depicted; it is bounded by the xy-plane, the yz-plane, and the planes with equations $x - y - 1 = 0$ and $2y + z = 0$.

Determine the limits of integration when changing the order of integration as

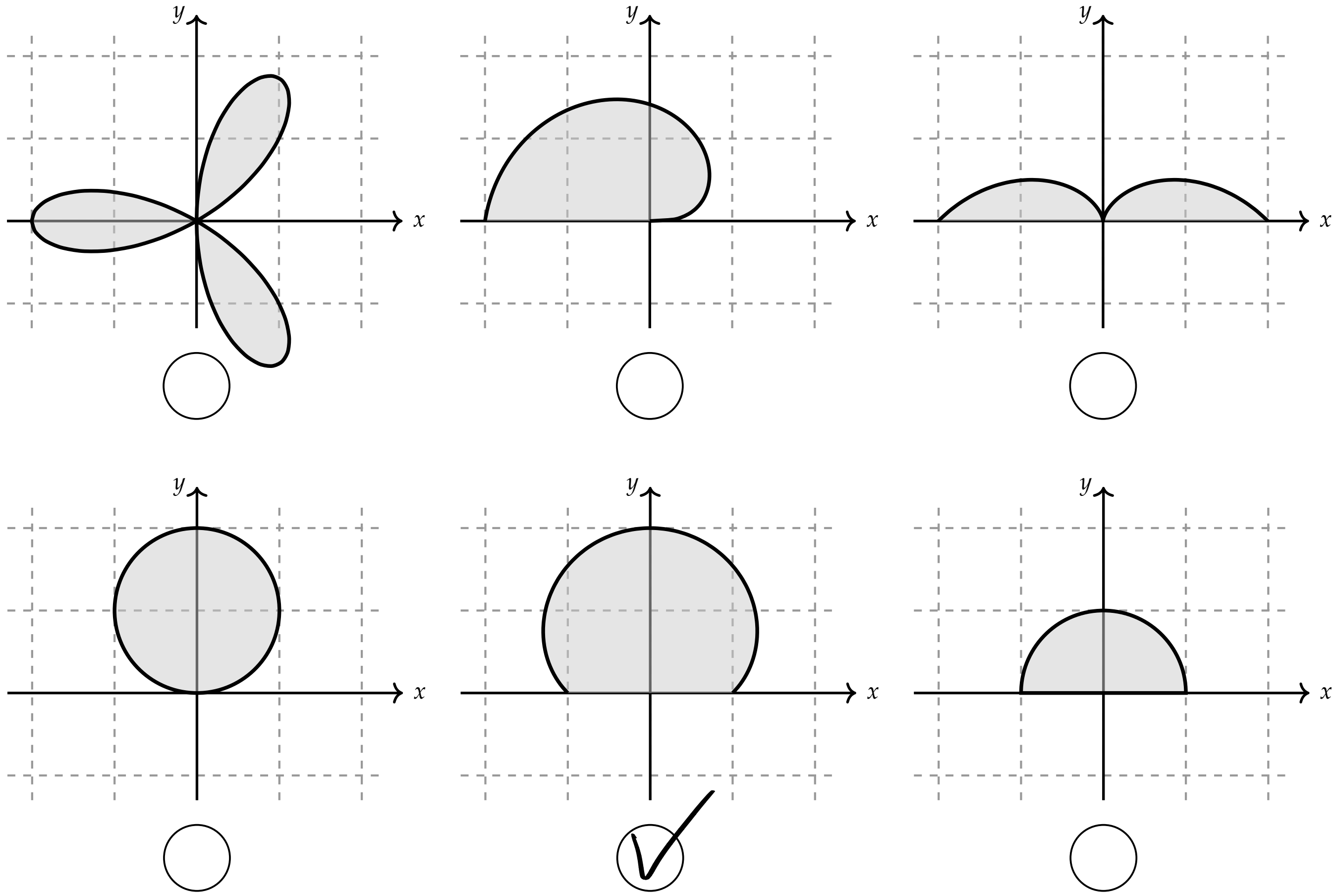
$$\int_0^2 \int_{-1}^{-z/2} \int_0^{1+y} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{x-1}}^{\boxed{0}} \int_{\boxed{0}}^{\boxed{-2y}} f(x, y, z)\, dz\, dy\, dx.$$

Scratch Space

Question 3. The double integral $\iint_R xy^2 \, dA$ has the form $\int_0^{\pi} \int_0^{1+\sin(\theta)} ?? \, dr \, d\theta$ when converted into polar coordinates.

(a) **(2 points)** Mark the box of the picture below which depicts the region R in the xy-plane.



(b) **(3 points)** Fill in the missing integrand to convert this integral to polar coordinates. Do **not** compute the integral!

$$\iint_R xy^2 \, dA = \int_0^{\pi} \int_0^{1+\sin\theta} \boxed{r^4 \sin^2\theta \cos\theta} \, dr \, d\theta$$

Scratch Space

$xy^2 dA = r\cos\theta \; r^2 \sin^2\theta \; r \; dr \, d\theta$

Question 4. **(7 points)** Let R be the region in $\mathbb{R}^3$ that is inside the sphere $x^2 + y^2 + z^2 = 9$ and below the cone $z = \sqrt{x^2 + y^2}$. Convert the triple integral $\iiint_R x\, dV$ into spherical coordinates. Do **not** compute the integral!

$$x\, dV = \rho \sin\phi \cos\theta \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$\iiint_R x\, dV = \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{\pi/4}}^{\boxed{\pi}} \int_{\boxed{0}}^{\boxed{3}} \boxed{\rho^3 \sin^2\phi \cos\theta}\, d\rho\, d\phi\, d\theta.$$

Note: The order of integration is already determined.

Scratch Space

Question 5. Let D be the square $\{(u,v) \mid 0 \le u \le 2 \text{ and } 0 \le v \le 2\}$.
Let $x(u,v) = v^2 - u^2$ and $y(u,v) = uv$. Consider the transformation $T(u,v) = \big(x(u,v), y(u,v)\big) = (v^2 - u^2, uv)$.
The transformation T satisfies:

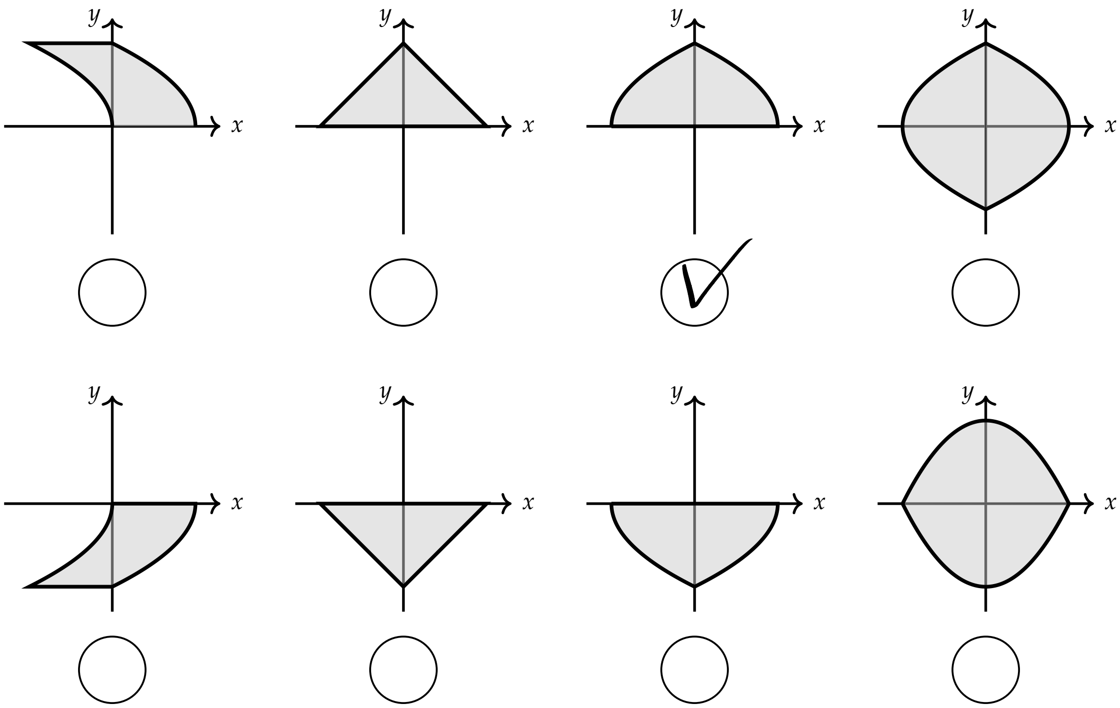
$T(u,0) = (-u^2, 0)$

$T(u,2) = (4 - u^2, 2u)$

$T(0,v) = (v^2, 0)$

$T(2,v) = (v^2 - 4, 2v)$

(a) **(2 points)** Mark the picture which depicts the image $T(D)$.



(b) **(2 points)** Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -2u & v \\ 2v & u \end{vmatrix} = -2u^2 - 2v^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{-2u^2 - 2v^2}$$

(c) **(3 points)** Set up an iterated integral computing the area of $T(D)$. Do **not** compute the integral!

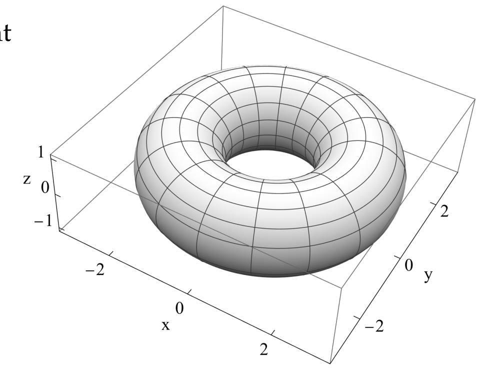
$$\text{Area}(T(D)) = \int_{\boxed{0}}^{\boxed{2}} \int_{\boxed{0}}^{\boxed{2}} \boxed{2u^2 + 2v^2}\, du\, dv$$

Question 6. Let T be the surface parametrized by

$$\mathbf{r}(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u), \quad \text{for } u, v \text{ in } [0, 2\pi].$$

The surface T can be obtained by revolving the circle $(x-2)^2 + z^2 = 1$ around the z-axis.

(a) **(4 points)** Find the equation for the tangent plane to T at the point $P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right) = \mathbf{r}(\pi, 3\pi/4)$.



A normal vector is given by $\vec{r}_u(\pi, \frac{3\pi}{4}) \times \vec{r}_v(\pi, \frac{3\pi}{4})$

$\vec{r}_u = \langle -\sin u \cos v, -\sin u \sin v, \cos u \rangle$

$\rightsquigarrow \vec{r}_u(\pi, \frac{3\pi}{4}) = \langle 0, 0, -1 \rangle$

$\vec{r}_v = \langle -(2+\cos u)\sin v, (2+\cos u)\cos v, 0 \rangle$

$\rightsquigarrow \vec{r}_v(\pi, \frac{3\pi}{3}) = \langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 0 \rangle$

$$\rightsquigarrow \vec{r}_u(\pi, \tfrac{3\pi}{4}) \times \vec{r}_v(\pi, \tfrac{3\pi}{4}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & -1 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{vmatrix} = \langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \rangle = \vec{n}$$

Plane equation: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\rightsquigarrow \sqrt{2}\langle -\frac{1}{2}, \frac{1}{2}, 0 \rangle \cdot \langle x + \frac{\sqrt{2}}{2}, y - \frac{\sqrt{2}}{2}, z \rangle = 0 \quad \rightsquigarrow -x - \frac{\sqrt{2}}{2} + y - \frac{\sqrt{2}}{2} = 0$

Tangent plane $\boxed{x - y = -\sqrt{2}}$

(b) **(2 points)** Which of the following statements is true? You do **not** need to use the parameterization to calculate the integral.

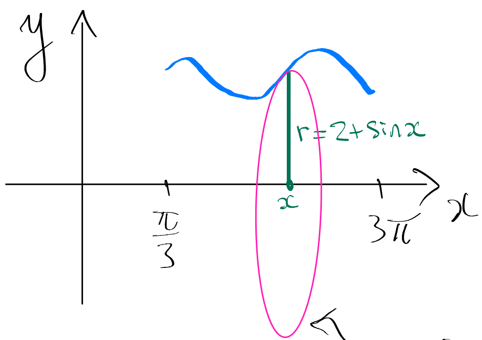
$\iint_T x \, dS < \iint_T z \, dS$ ○

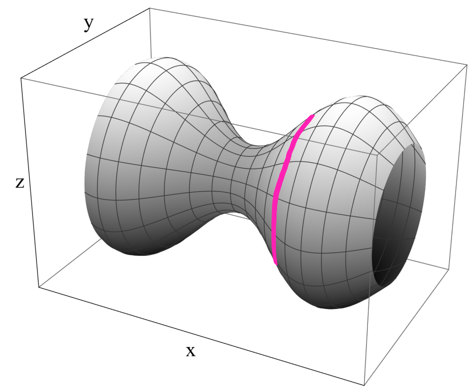
$\iint_T x \, dS = \iint_T z \, dS$ ✓

$\iint_T x \, dS > \iint_T z \, dS$ ○

Scratch Space

Question 7. **(3 points)** A surface of revolution is obtained by rotating the curve $y = 2 + \sin x$ around the x-axis. Let S be the portion of that surface that lies between the planes $x = \pi/3$ and $x = 3\pi$.
Parameterize S with a function $\mathbf{r}(u, v)$. Be sure to specify the domain D of your parameterization.





circle in y-z-plane of radius r = 2 + sin x

parametrize by angle v

$\mathbf{r}(u, v) =$ $\langle \; u \;, (2+\sin u)\cos v, (2+\sin u)\sin v \rangle$

$D =$ $\{(u, v) \mid \frac{\pi}{3} \le u \le 3\pi \;, \; 0 \le v \le 2\pi \}$

Scratch Space