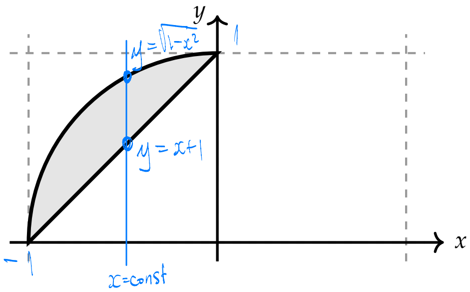
Question 1.

Let R be the depicted region above the line $x - y + 1 = 0$ and inside unit circle centered at the origin.



(a) **(4 points)** Find the bounds of integration for $\iint_R 2y\ dA$ as an iterated integral

$$\iint_R 2y\ dA = \int_{\boxed{-1}}^{\boxed{0}} \int_{\boxed{x+1}}^{\boxed{\sqrt{1-x^2}}} 2y\ dy\ dx.$$

Note: The order of integration is already determined.

(b) **(2 points)** Evaluate the integral $\iint_R 2y\ dA$ as set up in part (a).

$$\iint_R 2y\,dA = \int_{-1}^{0}\int_{x+1}^{\sqrt{1-x^2}} 2y\,dy\,dx = \int_{-1}^{0} y^2 \Big|_{y=x+1}^{\sqrt{1-x^2}} dx =$$

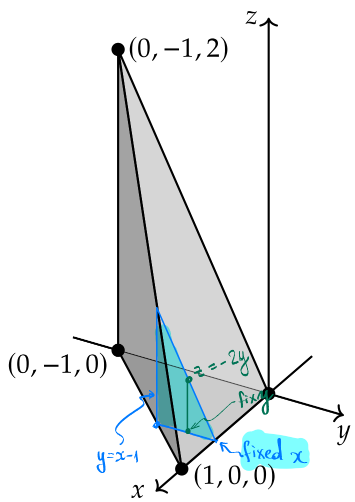
$$= \int_{-1}^{0} (1-x^2) - (x+1)^2\,dx = \int_{-1}^{0} -2x^2 - 2x\,dx =$$

$$= -\frac{2x^3}{3} - x^2 \Big|_{-1}^{0} = -\frac{2}{3} + 1$$

$$\iint_R 2y\ dA = \boxed{1/3}$$

Scratch Space

Question 2. **(6 points)**



Consider the triple integral $\int_0^2 \int_{-1}^{-z/2} \int_0^{1+y} f(x, y, z)\, dx\, dy\, dz$. Its region of integration is depicted; it is bounded by the xy-plane, the yz-plane, and the planes with equations $x - y - 1 = 0$ and $2y + z = 0$.

Determine the limits of integration when changing the order of integration as

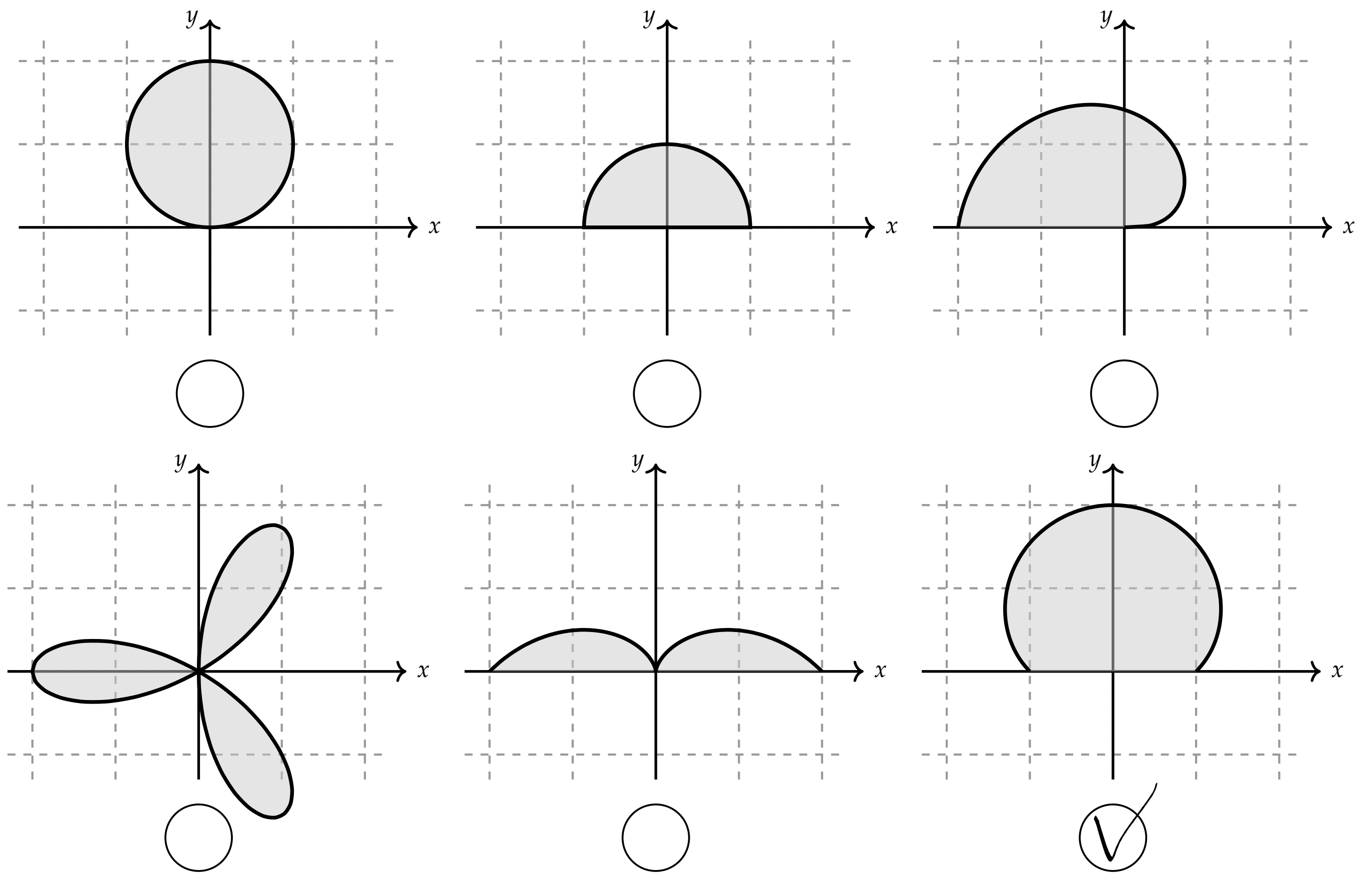
$$\int_0^2 \int_{-1}^{-z/2} \int_0^{1+y} f(x, y, z)\, dx\, dy\, dz$$

$$= \int_{\boxed{0}}^{\boxed{1}} \int_{\boxed{x-1}}^{\boxed{0}} \int_{\boxed{0}}^{\boxed{-2y}} f(x, y, z)\, dz\, dy\, dx.$$

Scratch Space

Question 3. The double integral $\iint_R x^2 y \, dA$ has the form $\int_0^{\pi} \int_0^{1+\sin(\theta)} ?? \, dr \, d\theta$ when converted into polar coordinates.

(a) **(2 points)** Mark the box of the picture below which depicts the region R in the xy-plane.



(b) **(3 points)** Fill in the missing integrand to convert this integral to polar coordinates. Do **not** compute the integral!

$$\iint_R x^2 y \, dA = \int_0^{\pi} \int_0^{1+\sin\theta} \boxed{r^4 \sin\theta \cos^2\theta} \, dr \, d\theta$$

Scratch Space

$x^2 y \, dA = r^2 \cos^2\theta \; r \sin\theta \; r \, dr \, d\theta$

Question 4. **(7 points)** Let R be the region in $\mathbb{R}^3$ that is inside the sphere $x^2 + y^2 + z^2 = 4$ and below the cone $z = \sqrt{x^2 + y^2}$. Convert the triple integral $\iiint_R y\, dV$ into spherical coordinates. Do **not** compute the integral!

$$y\, dV = \rho \sin\phi \sin\theta \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$\iiint_R y\, dV = \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{\pi/4}}^{\boxed{\pi}} \int_{\boxed{0}}^{\boxed{2}} \boxed{\rho^3 \sin^2\phi \sin\theta}\; d\rho\, d\phi\, d\theta.$$

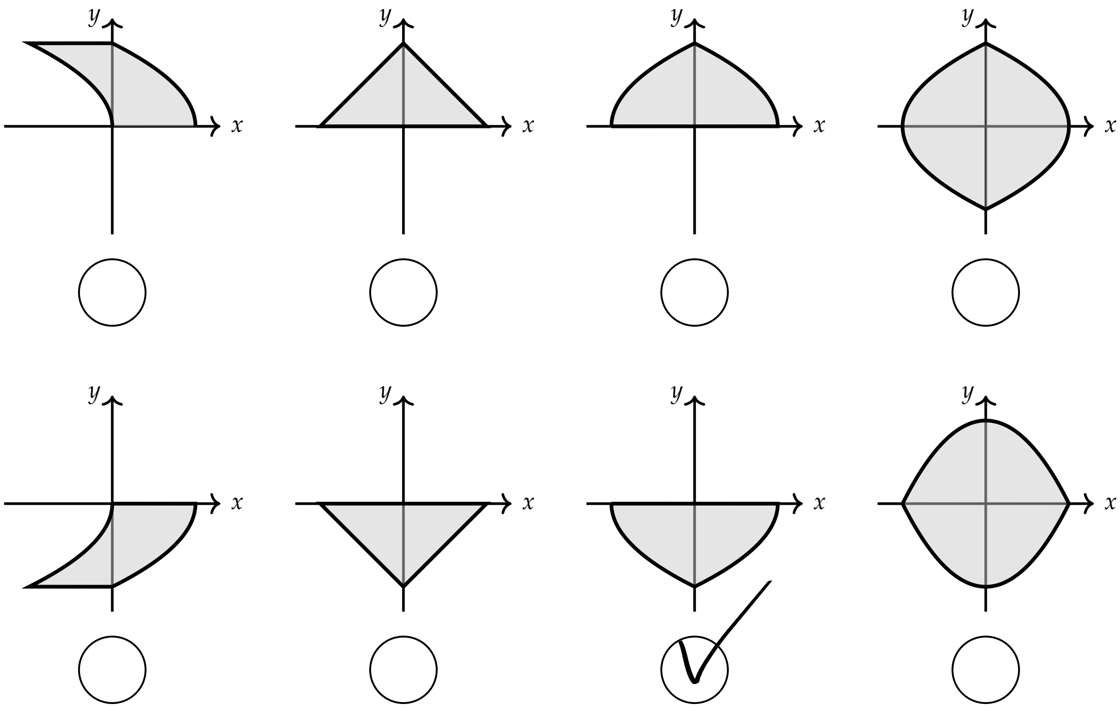
Note: The order of integration is already determined.

Scratch Space

Question 5. Let D be the square $\{(u,v) | -3 \le u \le 0 \text{ and } 0 \le v \le 3\}$.
Let $x(u,v) = v^2 - u^2$ and $y(u,v) = uv$. Consider the transformation $T(u,v) = \big(x(u,v), y(u,v)\big) = (v^2 - u^2, uv)$.
The transformation T satisfies:

$$T(u,0) = (-u^2, 0)$$
$$T(u,3) = (9 - u^2, 3u)$$
$$T(0,v) = (v^2, 0)$$
$$T(-3,v) = (v^2 - 9, -3v)$$

(a) **(2 points)** Mark the picture which depicts the image $T(D)$.



(b) **(2 points)** Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -2u & v \\ 2v & u \end{vmatrix} = -2u^2 - 2v^2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \boxed{-2u^2 - 2v^2}$$

(c) **(3 points)** Set up an iterated integral computing the area of $T(D)$. Do **not** compute the integral!

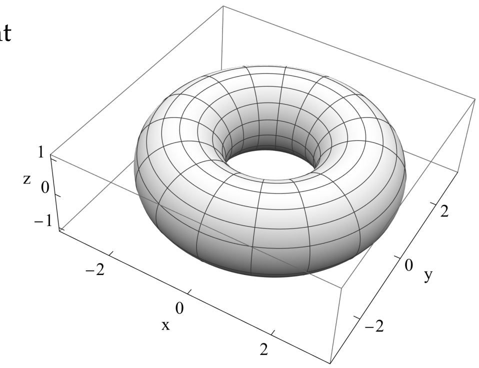
$$\text{Area}(T(D)) = \int_{\boxed{0}}^{\boxed{3}} \int_{\boxed{-3}}^{\boxed{0}} \boxed{2u^2 + 2v^2} \, du \, dv$$

Question 6. Let T be the surface parametrized by

$$\mathbf{r}(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u), \quad \text{for } u, v \text{ in } [0, 2\pi].$$

The surface T can be obtained by revolving the circle $(x-2)^2 + z^2 = 1$ around the z-axis.

(a) **(4 points)** Find the equation for the tangent plane to T at the point $P = \left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0\right) = \mathbf{r}(0, \pi/6)$.



A normal vector is given by $\vec{r}_u(0,\frac{\pi}{6}) \times \vec{r}_v(0,\frac{\pi}{6})$

$\vec{r}_u = \langle -\sin u \cos v, -\sin u \sin v, \cos u \rangle$

$\rightsquigarrow \vec{r}_u(0,\frac{\pi}{6}) = \langle 0, 0, 1 \rangle$

$\vec{r}_v = \langle -(2+\cos u)\sin v, (2+\cos u)\cos v, 0 \rangle$

$\rightsquigarrow \vec{r}_v(0,\frac{\pi}{6}) = \langle -\frac{3}{2}, \frac{3\sqrt{3}}{2}, 0 \rangle$

$$\rightsquigarrow \vec{r}_u(0,\tfrac{\pi}{6}) \times \vec{r}_v(0,\tfrac{\pi}{6}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ -\frac{3}{2} & \frac{3\sqrt{3}}{2} & 0 \end{vmatrix} = \left\langle -\frac{3\sqrt{3}}{2}, -\frac{3}{2}, 0 \right\rangle = \vec{n}$$

Plane equation: $\vec{n} \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$

$\rightsquigarrow -\frac{3}{2}\langle 3, \sqrt{3}, 0\rangle \cdot \langle x - \frac{3\sqrt{3}}{2}, y - \frac{3}{2}, z \rangle = 0$

$\rightsquigarrow 3x + \sqrt{3}y = 6\sqrt{3}$

or $\sqrt{3}x + y = 6$

Tangent plane $\boxed{\sqrt{3}x + y = 6}$

(b) **(2 points)** Which of the following statements is true? You do **not** need use the parameterization to calculate the integral.

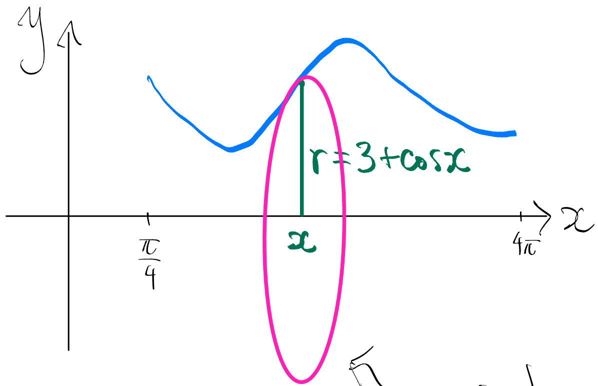
○ $\iint_T y\, dS > \iint_T z\, dS$

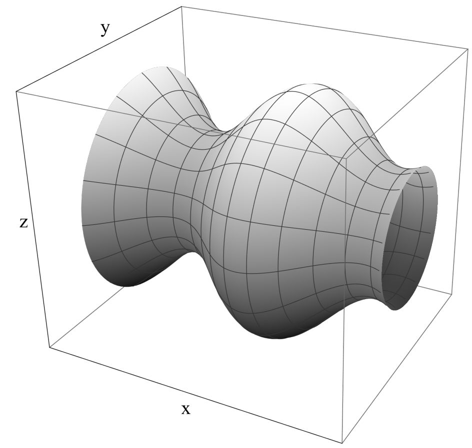
○ $\iint_T y\, dS < \iint_T z\, dS$

✓ $\iint_T y\, dS = \iint_T z\, dS$

Scratch Space

Question 7. **(3 points)** A surface of revolution is obtained by rotating the curve $y = 3 + \cos x$ around the x-axis. Let S be the portion of that surface that lies between the planes $x = \pi/4$ and $x = 4\pi$.
Parameterize S with a function $\mathbf{r}(u, v)$. Be sure to specify the domain D of your parameterization.





circle in y-z-plane
of radius $r = 3+\cos x$
parametrize by angle v

$\mathbf{r}(u, v) = \boxed{\langle \; u \; , (3+\cos u)\cos v \; , (3+\cos u)\sin v \rangle}$

$D = \boxed{\{(u, v) \mid \frac{\pi}{4} \le u \le \pi \; , \; 0 \le v \le 2\pi \}}$

Scratch Space