Question 1.

Let A be the oriented line segment from $(0,-4)$ to $(0,0)$, B be the oriented line segment from $(0,0)$, to $(4,0)$, and C the oriented line segment from $(4,0)$ to $(0,-4)$.

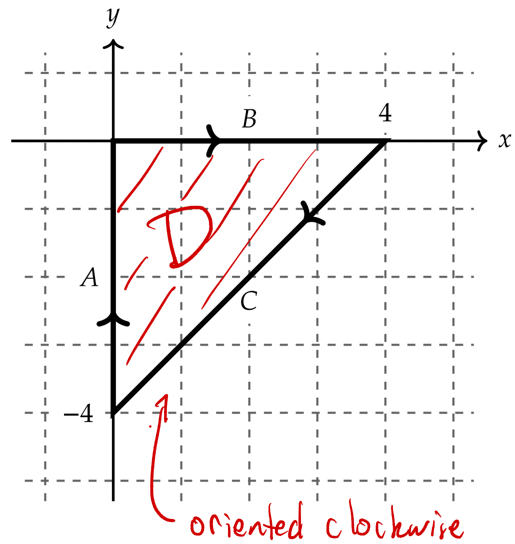
The vector field $\mathbf{F} = \langle P, Q \rangle$ satisfies

$Q_y = P_x$

$Q_x = P_y + 2$.

Suppose that $\int_B \mathbf{F} \cdot d\mathbf{r} = -4$ and $\int_A \mathbf{F} \cdot d\mathbf{r} = -2$.

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ and mark your answer. Show your work. **(5 points)**



Green's Theorem:

counterclockwise orientation

$$\iint_D Q_x - P_y \, dA = \int_{\partial D} \vec{F} \cdot d\vec{r} = -\int_{A+B+C} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = -\iint_D Q_x - P_y \, dA - \int_A \vec{F} \cdot d\vec{r} - \int_B \vec{F} \cdot d\vec{r} = -\iint_D 2 \, dA - (-2) - (-4)$$

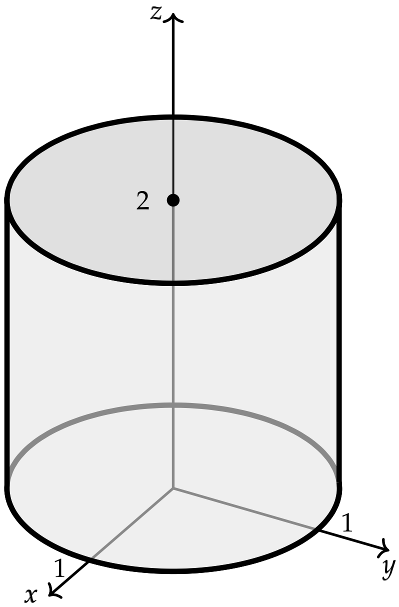
$$= -2(\text{area}(D)) + 2 + 4 = -16 + 2 + 4 = -10$$

$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $\boxed{-10}$

Scratch Space

Question 2.

Let $\mathbf{H} = \langle 6x + 3yz^2, -y, z + 2x\rangle$. Let E be the surface of the cylinder of radius 1, where $0 \le z \le 2$, oriented outwards, that is, with an outward pointing unit normal vector. Use the divergence theorem to compute the flux of $\mathbf{H}$ through E. **(5 points)**



Divergence theorem :

$$\iint_E \vec{H}\cdot d\vec{S} = \iiint_{\text{cylinder}} \operatorname{div}\vec{H}\, dV = \iiint_{\text{cylinder}} 6-1+1\, dV$$

$$= 6\left(\begin{matrix}\text{Volume}\\ \text{of}\\ \text{cylinder}\end{matrix}\right) = 6(\pi\cdot 1\cdot 2) = 12\pi$$

flux = $\boxed{12\pi}$

Question 3.

Find real numbers a and b so that $\mathbf{F}(x, y, z) = \langle e^{x^2} + 3az,\ 3zy,\ 9x + 3by^2\rangle$ is conservative. **(4 points)**

$\vec{F}$ conservative $\Longleftrightarrow$ curl $\vec{F} = \vec{0}$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2}+3az & 3zy & 9x+3by^2 \end{vmatrix} = \langle 6by-3y, -(9-3a), 0 \rangle$$

need $9-3a=0 \Rightarrow a=3$

$6by-3y=0 \Rightarrow b=1/2$

$a = \boxed{3}$

$b = \boxed{1/2}$

Scratch Space

Question 4.

Let S be the surface which is the portion of the graph of $y = z^2$ where $0 \le x \le 1$ and $-1 \le z \le 0$, oriented upwards, that is, with upward pointing unit normal vector. Let $\mathbf{F} = \langle 3yz, 3x^2 + 2y, z\rangle$. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

(6 points)

parameterize:

$\vec{r}(u,v) = \langle u, v^2, v \rangle$ $\quad$ $0 \le u \le 1$, $-1 \le v \le 0$

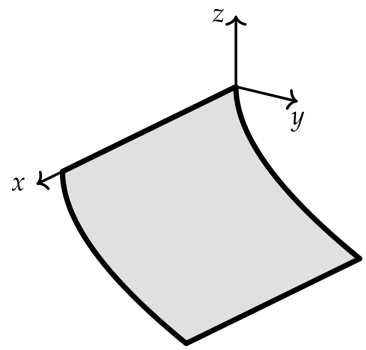
$\vec{r}_u = \langle 1, 0, 0 \rangle$

$\vec{r}_v = \langle 0, 2v, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, -1, 2v \rangle$

downward for $v<0$

correction due to wrong direction of $\vec{r}_u \times \vec{r}_v$



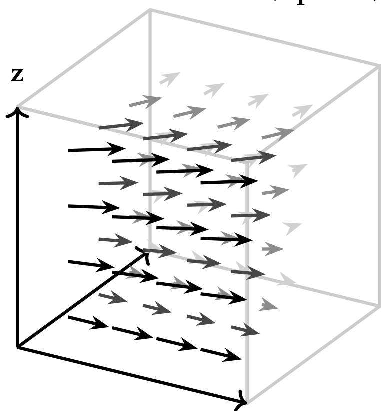
$$\iint_S \vec{F}\cdot d\vec{S} = -\int_{-1}^{0}\int_{0}^{1} \langle 3v^3, 3u^2+2v^2, v\rangle \cdot \langle 0,-1,2v\rangle \, du\, dv$$

$$= -\int_{-1}^{0}\int_{0}^{1} -3u^2 - 2v^2 + 2v^2 \, du\, dv = -\int_{-1}^{0} u^3 \Big|_0^1 = -1$$

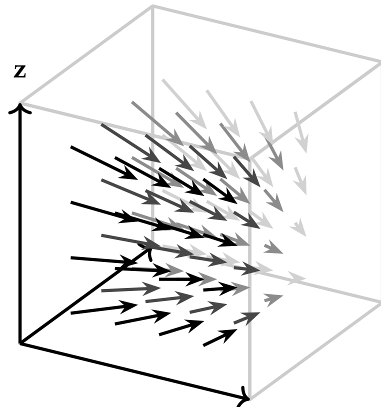
$\iint_S \mathbf{F} \cdot d\mathbf{S} =$ $\boxed{-1}$

Scratch Space

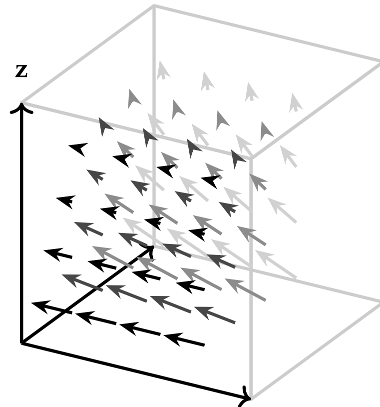
Question 5.
Consider the following vector fields. The x and z axes are labeled; the remaining unlabeled axis is the y-axis. For each part mark the best answer. **(5 points)**



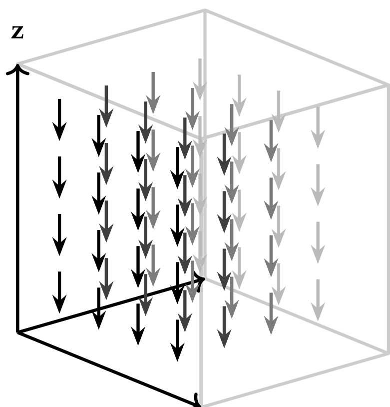
A



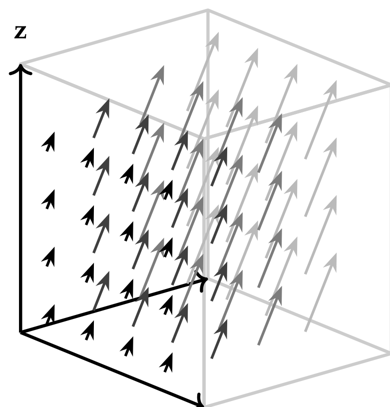
B



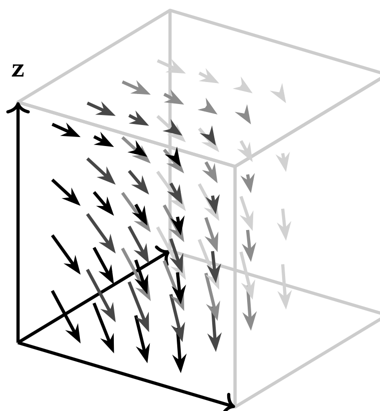
C



D



E



F

(a) Exactly one of these vector fields has nonzero divergence. Which one?

A ◯ **B** ◯ **C** ◯ **D** ◯ **E** ◯ **F** ◯

What is the sign of the divergence of the vector field you just marked?

◯ positive ◯ zero ◯ negative ◯ positive at some points and negative at some points

(b) Is the vector field you marked in part (a) equal to curl(**G**) for some vector field **G**?

◯ Yes ◯ No

(c) curl(**F**) is constant and is one of the following. What is its value?

−i ◯ **−j** ◯ **−k** ◯ 0 ◯ **i** ◯ **j** ◯ **k** ◯

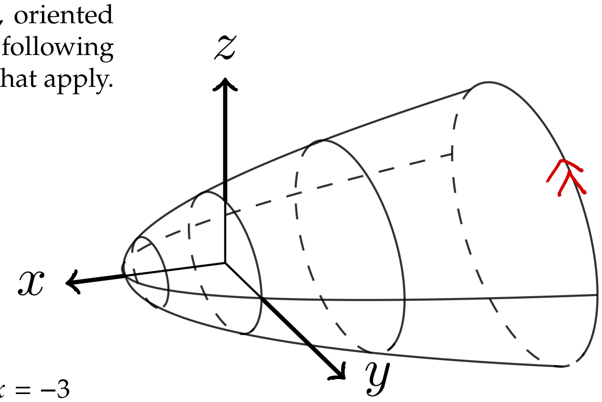
Question 6. Let $\mathbf{F} = \langle e^{x^2} + 3x^2y,\ 2x + y,\ z - 2x + y\rangle$.

(a) Compute curl(**F**). Mark your answer. **(2 points)**

$\langle 0, 2x+y, 0\rangle$	$\langle 1, -2, 2-3x^2\rangle$	$\langle -1, -2, -2+3x^2\rangle$	0	$\langle 1, 2, 2-3x^2\rangle$	$\langle -1, 2, -2+3x^2\rangle$	$\langle 0, -2x-y, 0\rangle$
○	○	○	○	⊗	○	○

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2}+3x^2y & 2x+y & z-2x+y \end{vmatrix} = \langle 1, -(-2-0), 2-3x^2\rangle$$

(b) Let S be the portion of the graph of $x = 1 - y^2 - z^2$ with $x \geq -3$, oriented outwards, that is, with outward pointing unit normal. Which of the following integrals have value equal to the flux of curl(**F**) through S? Mark all that apply. **(4 points)**



- [x] $\int_0^{2\pi} \mathbf{F}(\overset{-3}{\cancel{-2}}, \cos(t), \sin(t)) \cdot \langle 0, -\sin(t), \cos(t)\rangle\, dt$
- [ ] $\int_0^{2\pi} \mathbf{F}(\overset{-3}{\cancel{-2}}, \cos(t), \sin(t)) \cdot \langle \overset{-3}{\cancel{-2}}, \cos(t), \sin(t)\rangle\, dt$
- [ ] $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \langle 1, 0, 0\rangle\, dS$
- [x] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle 1, 0, 0\rangle\, dS$, where D is the disc $y^2 + z^2 \leq 4$ with $x = -3$
- [ ] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle 1, 0, 0\rangle\, dS$, where D is the disc $y^2 + z^2 \leq 1$ with $x = 0$
- [ ] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle -1, 0, 0\rangle\, dS$, where D is the disc $y^2 + z^2 \leq 4$ with $x = -3$
- [ ] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle -1, 0, 0\rangle\, dS$, where D is the disc $y^2 + z^2 \leq 1$ with $x = 0$

(c) Use one of the integrals you chose in part (b) to compute the flux of curl(**F**) through S. **(4 points)**

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \langle 1, 2, 2-3x^2\rangle \cdot \langle 1, 0, 0\rangle\, dS$$

$$= \iint_D dS = \operatorname{area}(D) = 4\pi$$

Flux = 4π

Question 7.

Let $\mathbf{F} = \langle P, Q \rangle$ where

$P(x,y) = \dfrac{2y}{x^2+y^2}$ and $Q(x,y) = 3 + \dfrac{-2x}{x^2+y^2}$.

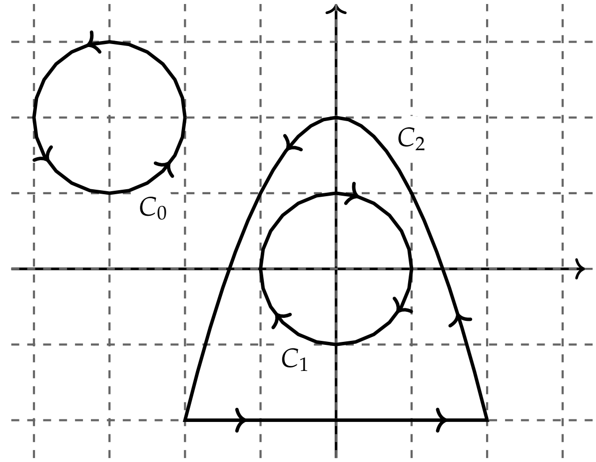
These satisfy

$$P_x = \frac{-4xy}{(x^2+y^2)^2}$$

$$P_y = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$Q_y = \frac{4xy}{(x^2+y^2)^2}$$



(a) Let C_0 be the circle of radius 1 centered at $(-3,2)$ and oriented counterclockwise. Compute $\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

-6π -4π $3-2\pi$ 0 $3+2\pi$ 4π 6π

(b) Let C_1 be the circle of radius 1 centered at $(0,0)$ and oriented clockwise. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

-6π -4π $3-2\pi$ 0 $3+2\pi$ 4π 6π

(c) Let C_2 be the closed curve consisting of the line segment from $(-2,-2)$ to $(2,-2)$ and the graph of $y = 2 - x^2$, oriented as displayed.
Which is the correct answer? **(1 point)**

$\left|\int_{C_1} \mathbf{F}\cdot d\mathbf{r}\right| < \left|\int_{C_2} \mathbf{F}\cdot d\mathbf{r}\right|$ $-\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ $\left|\int_{C_1} \mathbf{F}\cdot d\mathbf{r}\right| > \left|\int_{C_2} \mathbf{F}\cdot d\mathbf{r}\right|$

Scratch Space

parameterize C_1

$\vec{r} = \langle \sin t, \cos t \rangle \quad 0 \le t \le 2\pi$

$\vec{r}\,' = \langle \cos t, -\sin t \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{2\cos t}{\cos^2 t + \sin^2 t},\ 3 - \frac{2\sin t}{\cos^2 t + \sin^2 t} \right\rangle \cdot \langle \cos t, -\sin t \rangle \, dt$$

$$= \int_0^{2\pi} 2\cos^2 t - 3\sin t + 2\sin^2 t \, dt = \int_0^{2\pi} 2 - 3\sin t \, dt = 4\pi$$