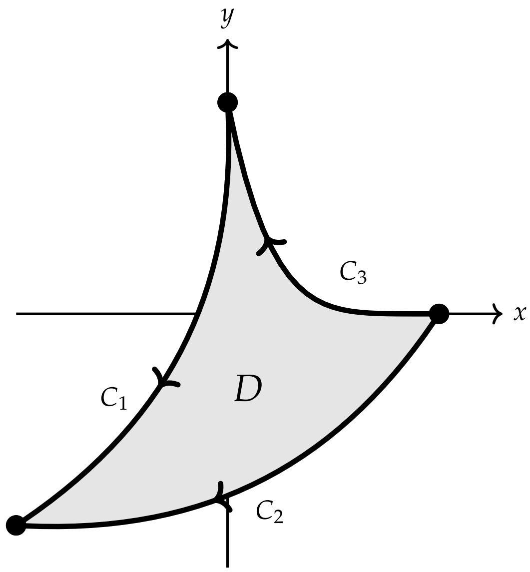
Question 1.
Let C_1 be the displayed path from $(0,1)$ to $(-1,-1)$, C_2 the displayed path from $(1,0)$ to $(-1,-1)$, and C_3 the displayed path from $(1,0)$ to $(0,1)$. Let D be the region bounded by the curves C_1, C_2, and C_3 and consider the vector field $\mathbf{F} = \langle 2-2x+4y, 7x-y \rangle$. We know that

P Q

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 5,$$
$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 1, \text{ and}$$
$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = -1.$$



Use Green's theorem and the vector field **F** to compute the area of D. Mark your answer. Show your work. **(5 points)**

$$\iint_D Q_x - P_y \, dA = \iint_D 7-4 \, dA = \iint_D 3 \, dA = 3\cdot \text{Area}(D)$$

By Green's Theorem:

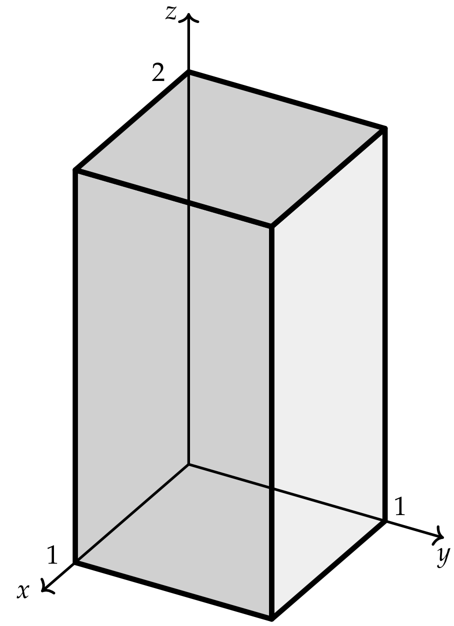
$$\iint_D Q_x - P_y \, dA = \int_{\partial D} \vec{F}\cdot d\vec{r} = \int_{C_1 - C_2 + C_3} \vec{F}\cdot d\vec{r} = 5-1-1 = 3$$

$$\Rightarrow \text{Area}(D) = \frac{3}{3} = 1$$

Area(D) = $\boxed{1}$

Scratch Space

Question 2.
Let $\mathbf{H} = \langle 3x + 2z^3, 5, 2z - y\rangle$. Let E be the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$, oriented outwards, that is, with an outward pointing unit normal vector. Use the divergence theorem to compute the flux of $\mathbf{H}$ through E. **(5 points)**



$\operatorname{div} \vec{H} = 3 + 2 = 5$

Divergence theorem:

$$\iint_E \vec{H} \cdot d\vec{S} = \iiint_{Box} \operatorname{div} \vec{H} \, dV = \iiint_{Box} 5 \, dV$$

$$= 5 \cdot \text{Volume}(Box)$$

$$= 5 \cdot 2 = 10$$

flux = $\boxed{10}$

Question 3.
Find real numbers a and b so that $\mathbf{F}(x, y, z) = \langle e^{x^2} + ay,\ 2x + zy,\ by^2\rangle$ is conservative. **(4 points)**

$\vec{F}$ conservative $\iff$ curl $\vec{F} = \vec{0}$

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2} + ay & 2x + zy & by^2 \end{vmatrix} = \langle 2by - y, 0, 2 - a\rangle$$

need $2by - y = 0 \Rightarrow b = 1/2$

$2 - a = 0 \Rightarrow a = 2$

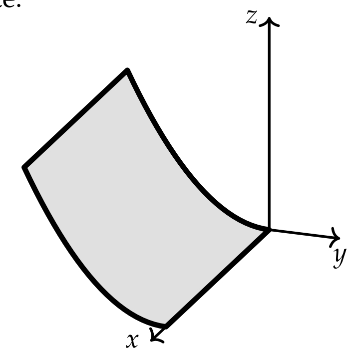
$a = \boxed{2}$

$b = \boxed{1/2}$

Scratch Space

Question 4.
Let S be the surface which is the portion of the graph of $z = y^2$ where $0 \le x \le 1$ and $-1 \le y \le 0$, oriented in the direction of the positive y-axis, that is, with unit normal vector having non-negative second coordinate.

Let $\mathbf{F} = \langle x^2 + z,\ y,\ z \rangle$. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. **(6 points)**



parameterize:

$\vec{r}(u,v) = \langle u, v, v^2 \rangle$ $\quad 0 \le u \le 1$, $-1 \le v \le 0$

$\vec{r}_u = \langle 1,0,0 \rangle$

$\vec{r}_v = \langle 0,1,2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, -2v, 1 \rangle$

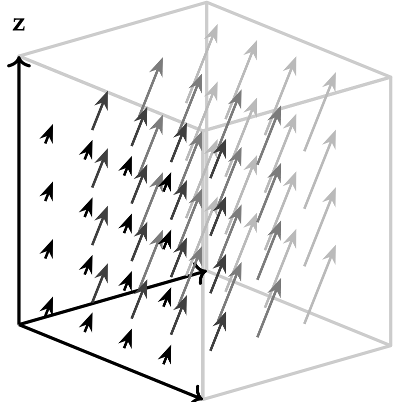
$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{0}\int_{0}^{1} \langle u^2+v^2, v, v^2 \rangle \cdot \langle 0, -2v, 1 \rangle \, du\,dv = \int_{-1}^{0}\int_{0}^{1} -2v^2 + v^2 \, du\,dv =$$

$$= \int_{-1}^{0}\int_{0}^{1} -v^2 \, du\,dv = -\frac{v^3}{3}\Big|_{-1}^{0} = -1/3$$

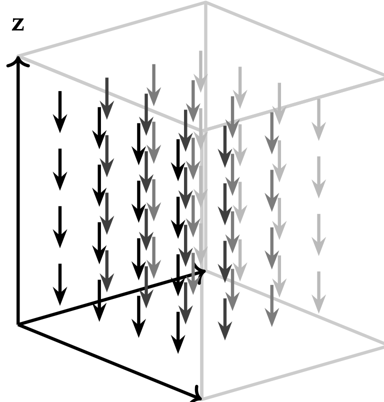
$\iint_S \mathbf{F} \cdot d\mathbf{S} = \boxed{-1/3}$

Scratch Space

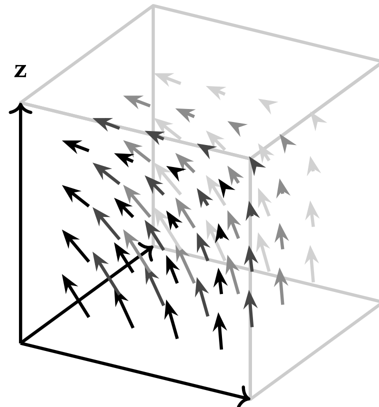
Question 5.
Consider the following vector fields. The x and z axes are labeled; the remaining unlabeled axis is the y-axis. For each part mark the best answer. **(5 points)**



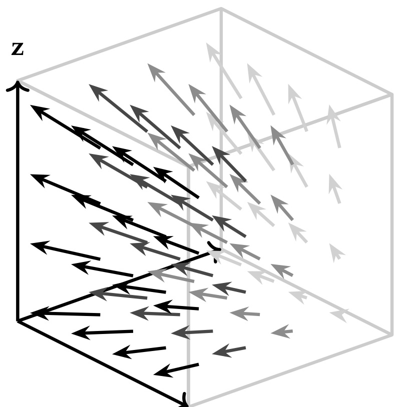
A



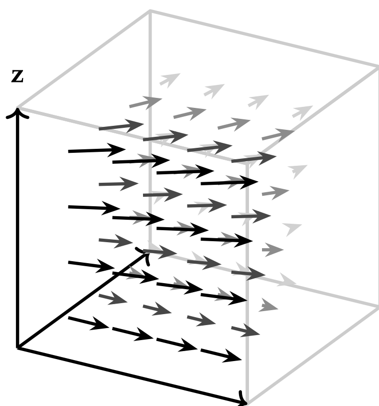
B



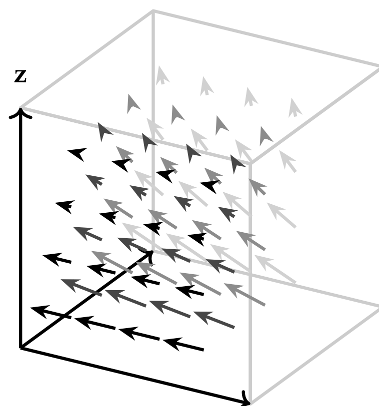
C



D



E



F

(a) Exactly one of these vector fields has nonzero divergence. Which one?

A	**B**	**C**	**D**	**E**	**F**
○	○	○	⊗	○	○

What is the sign of the divergence of the vector field you just marked?

⊗ positive ○ zero ○ negative ○ positive at some points and negative at some points

(b) Is the vector field you marked in part (a) equal to curl(**G**) for some vector field **G**?

○ Yes ⊗ No

(c) curl(**C**) is constant and is one of the following. What is its value?

$-\mathbf{i}$	$-\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
○	⊗	○	○	○	○	○

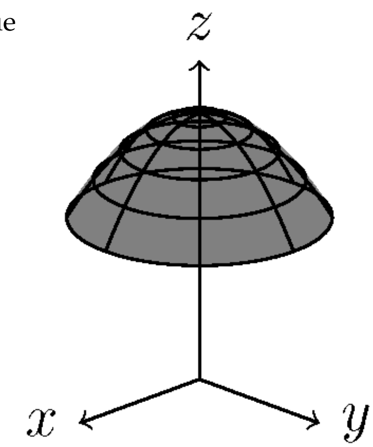
Question 6. Let $\mathbf{F} = \langle e^{x^2} + 2y,\ 4x,\ y^2z + x\rangle$.

(a) Compute curl($\mathbf{F}$). Mark your answer. **(2 points)**

$\langle 0,0,y^2\rangle$	$\langle 2yz,\ -1,\ 2\rangle$	$\langle -2yz,\ -1,\ -2\rangle$	0	$\langle 2yz,\ 1,\ 2\rangle$	$\langle -2yz,\ 1,\ -2\rangle$	$\langle 0,\ 0,\ -y^2\rangle$
○	⊗	○	○	○	○	○

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2}+2y & 4x & y^2z+x \end{vmatrix} = \langle 2yz - 0, -(1-0), 4-2\rangle$$

(b) Let S be the portion of the graph of $z = 4 - x^2 - y^2$ with $z \geq 3$, oriented upwards, that is, with upward pointing unit normal. Which of the following integrals have value equal to the flux of curl($\mathbf{F}$) through S? Mark all that apply. **(4 points)**



- [ ] $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \langle 0,\ 0,\ 1\rangle\, dS$
- [X] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle 0,\ 0,\ 1\rangle\, dS$, where D is the disc $x^2 + y^2 \leq 1$ with $z = 3$
- [ ] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle 0,\ 0,\ 1\rangle\, dS$, where D is the disc $x^2 + y^2 \leq 2$ with $z = 0$
- [ ] $\iint_D \operatorname{curl}(\mathbf{F}) \cdot \langle 0,\ 0,\ -1\rangle\, dS$, where D is the disc $x^2 + y^2 \leq 1$ with $z = 3$
- [X] $\int_0^{2\pi} \mathbf{F}(\cos(t),\ \sin(t),\ 3) \cdot \langle -\sin(t), \cos(t), 0\rangle\, dt$
- [ ] $\int_0^{2\pi} \mathbf{F}(\cos(t),\ \sin(t),\ 3) \cdot \langle \cos(t),\ \sin(t),\ 3\rangle\, dt$

(c) Use one of the integrals you chose in part (b) to compute the flux of curl($\mathbf{F}$) through S. **(4 points)**

$$\iint_S \operatorname{curl}\vec{F}\, d\vec{S} = \iint_D \langle 2yz, -1, 2\rangle \cdot \langle 0,0,1\rangle\, dS = \iint_D 2\, dS$$

$$= 2 \cdot \text{Area}(D) = 2\pi$$

Flux = $\boxed{2\pi}$

Question 7.

Let $\mathbf{F} = \langle P, Q \rangle$ where

$P(x,y) = \dfrac{y}{x^2+y^2}$ and $Q(x,y) = 3 + \dfrac{-x}{x^2+y^2}$.

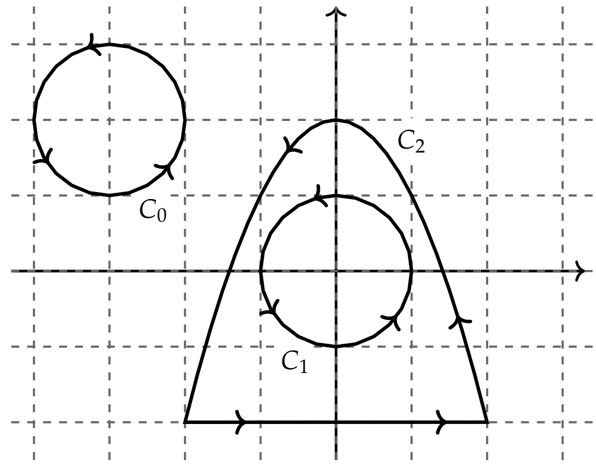
These satisfy

$$P_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$P_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q_y = \frac{2xy}{(x^2+y^2)^2}$$



(a) Let C_0 be the circle of radius 1 centered at $(-3, 2)$ and oriented counterclockwise. Compute $\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

3π	2π	$3+\pi$	0	$3-\pi$	-2π	-3π
○	○	○	⊗	○	○	○

(b) Let C_1 be the circle of radius 1 centered at $(0, 0)$ and oriented counterclockwise. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

3π	2π	$3+\pi$	0	$3-\pi$	-2π	-3π
○	○	○	○	○	⊗	○

(c) Let C_2 be the closed curve consisting of the line segment from $(-2, -2)$ to $(2, -2)$ and the graph of $y = 2 - x^2$, oriented as displayed.

Which is the correct answer? **(1 point)**

$\left\lvert\int_{C_1} \mathbf{F}\cdot d\mathbf{r}\right\rvert > \left\lvert\int_{C_2} \mathbf{F}\cdot d\mathbf{r}\right\rvert$	$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$	$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = -\int_{C_2} \mathbf{F}\cdot d\mathbf{r}$	$\left\lvert\int_{C_1} \mathbf{F}\cdot d\mathbf{r}\right\rvert < \left\lvert\int_{C_2} \mathbf{F}\cdot d\mathbf{r}\right\rvert$
○	⊗	○	○

Scratch Space

parameterize C_1

$\vec{r} = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}\,' = \langle -\sin t, \cos t \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{\sin t}{\cos^2 t + \sin^2 t},\ 3 \frac{-\cos t}{\cos^2 t + \sin^2 t} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t + 3\cos t - \cos^2 t \, dt = \int_0^{2\pi} -1 + 3\cos t \, dt = -2\pi$$