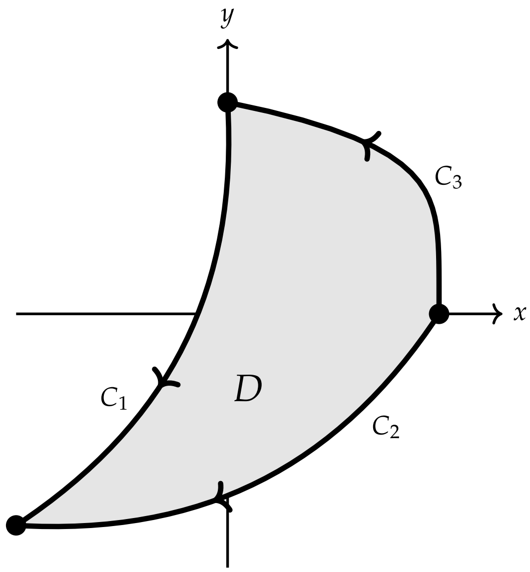
**Question 1.**

Let $C_1$ be the displayed path from $(0,1)$ to $(-1,-1)$, $C_2$ the displayed path from $(1,0)$ to $(-1,-1)$, and $C_3$ the displayed path from $(1,0)$ to $(0,1)$. Let $D$ be the region bounded by the curves $C_1$, $C_2$, and $C_3$ and consider the vector field $\mathbf{F} = \langle 1-4x+4y,\, 6x-3y\rangle$. We know that




$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = 3,$$ P Q

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 1, \text{ and}$$

$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = 2.$$

Use Green's theorem and the vector field **F** to compute the area of $D$. Mark your answer. Show your work. **(5 points)**

$$\iint_D Q_x - P_y \, dA = \iint_D 6-4 \, dA = \iint_D 2 \, dA = 2 \cdot \text{Area}(D)$$

By Green's Theorem:

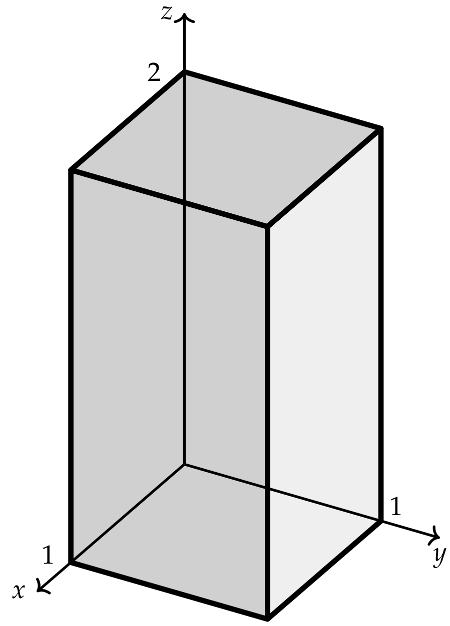
$$\iint_D Q_x - P_y \, dA = \int_{\partial D} \vec{F}\cdot d\vec{r} = \int_{C_1 - C_2 + C_3} \vec{F}\cdot d\vec{r} = 3 - 1 + 2 = 4$$

$$\Rightarrow \text{Area}(D) = \frac{4}{2} = 2$$

Area$(D)$ = $\boxed{2}$

**Scratch Space**

**Question 2.**
Let $\mathbf{H} = \langle 2x + 3z^2, -y, 3z + 2x \rangle$. Let $E$ be the surface of the box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, and $z = 2$, oriented outwards, that is, with an outward pointing unit normal vector. Use the divergence theorem to compute the flux of $\mathbf{H}$ through $E$. **(5 points)**




$$\text{div}\,\vec{H} = 2 - 1 + 3 = 4$$

Divergence theorem:

$$\iint_E \vec{H}\cdot d\vec{S} = \iiint_{Box} \text{div}\,\vec{H}\,dV = \iiint_{Box} 4\,dV$$

$$= 4\cdot \text{Volume(Box)}$$

$$= 4\cdot 2 = 8$$

flux = $\boxed{8}$

**Question 3.**
Find real numbers $a$ and $b$ so that $\mathbf{F}(x, y, z) = \langle e^{x^2} + az^2, 4z, 6xz + 2by \rangle$ is conservative. **(4 points)**

$\vec{F}$ conservative $\iff$ curl $\vec{F} = \vec{0}$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2}+az^2 & 4z & 6xz+2by \end{vmatrix} = \langle 2b-4, -(6z-2az), 0 \rangle$$

need $2b - 4 = 0 \Rightarrow b = 2$

$6 - 2a \Rightarrow a = 3$

$a = \boxed{3}$

$b = \boxed{2}$

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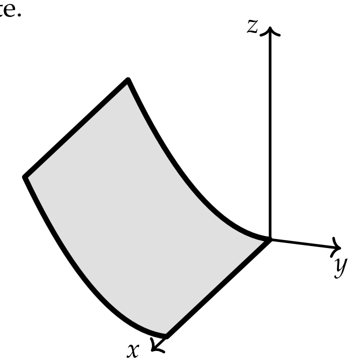
**Scratch Space**

**Question 4.**

Let $S$ be the surface which is the portion of the graph of $z = y^2$ where $0 \le x \le 1$ and $-1 \le y \le 0$, oriented in the direction of the positive $y$-axis, that is, with unit normal vector having non-negative second coordinate.

Let $\mathbf{F} = \langle z - x^2, -2y, -3z \rangle$. Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$. **(6 points)**




parameterize:

$\vec{r}(u,v) = \langle u, v, v^2 \rangle$  $0 \le u \le 1$, $-1 \le v \le 0$

$\vec{r}_u = \langle 1,0,0 \rangle$

$\vec{r}_v = \langle 0,1,2v \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 0, -2v, 1 \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{0} \int_0^1 \langle v^2 - u^2, -2v, -3v^2 \rangle \cdot \langle 0, -2v, 1 \rangle \, du\,dv = \int_{-1}^{0} \int_0^1 4v^2 - 3v^2 \, du\,dv =$$

$$= \int_{-1}^{0} \int_0^1 v^2 \, du\,dv = \frac{v^3}{3} \Big|_{-1}^{0} = 1/3$$

$\iint_S \mathbf{F} \cdot d\mathbf{S} =$ $\boxed{1/3}$

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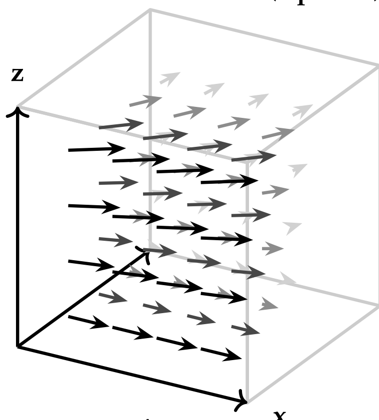
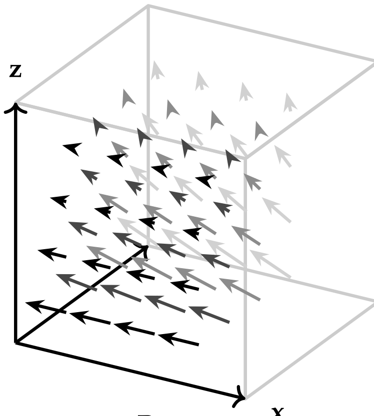
**Scratch Space**

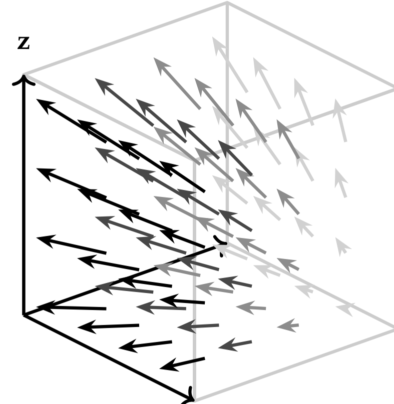
**Question 5.**
Consider the following vector fields. The $x$ and $z$ axes are labeled; the remaining unlabeled axis is the $y$-axis. For each part mark the best answer. **(5 points)**



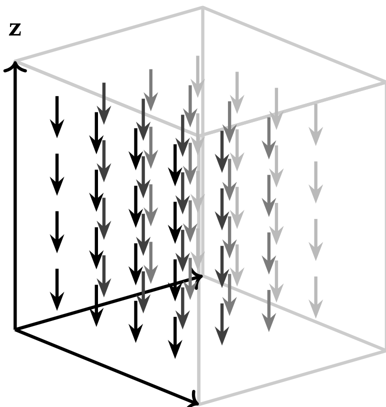

**A**



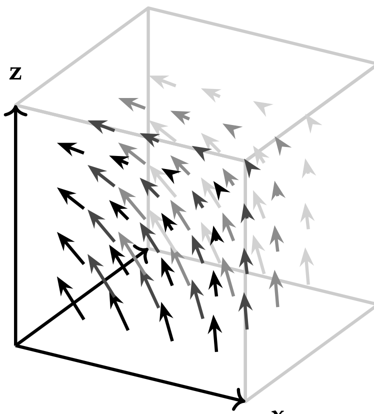

**B**



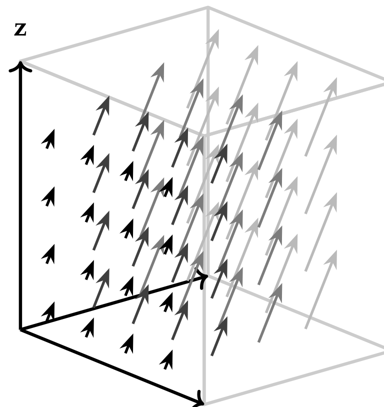

**C** x




**D**




**E**




**F**

(a) Exactly one of these vector fields has nonzero divergence. Which one?

**A** ○ **B** ○ **C** ○ **D** ○ **E** ○ **F** ○

What is the sign of the divergence of the vector field you just marked?

○ positive ○ zero ○ negative ○ positive at some points and negative at some points

(b) Is the vector field you marked in part (a) equal to curl(**G**) for some vector field **G**?

○ Yes ○ No

(c) curl(**E**) is constant and is one of the following. What is its value?

**i** ○ **j** ○ **k** ○ 0 ○ **−i** ○ **−j** ○ **−k** ○

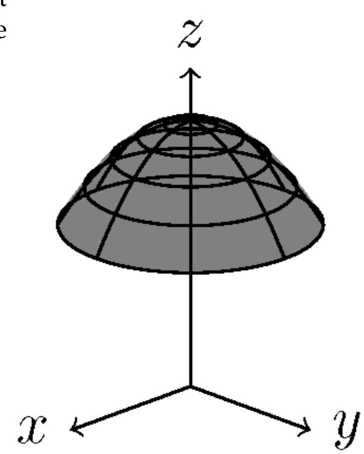
**Question 6.** Let $\mathbf{F} = \langle e^{x^2} + 2y + z,\ 5x,\ y^2 z\rangle$.

(a) Compute curl($\mathbf{F}$). Mark your answer. **(2 points)**

- [ ] $\langle 0, 0, y^2\rangle$
- [ ] $\langle 2yz, -1, 3\rangle$
- [ ] $\langle -2yz, -1, -3\rangle$
- [ ] $0$
- [x] $\langle 2yz, 1, 3\rangle$
- [ ] $\langle -2yz, 1, -3\rangle$
- [ ] $\langle 0, 0, -y^2\rangle$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{x^2}+2y+z & 5x & y^2 z \end{vmatrix} = \langle 2yz, -(0-1), 5-2\rangle$$

(b) Let $S$ be the portion of the graph of $z = 4 - x^2 - y^2$ with $z \geq 3$, oriented upwards, that is, with upward pointing unit normal. Which of the following integrals have value equal to the flux of curl($\mathbf{F}$) through $S$? Mark all that apply. **(4 points)**




- [x] $\int_0^{2\pi} \mathbf{F}\big(\cos(t), \sin(t), 3\big) \cdot \langle -\sin(t), \cos(t), 0\rangle\, dt$
- [ ] $\int_0^{2\pi} \mathbf{F}\big(\cos(t), \sin(t), 3\big) \cdot \langle \cos(t), \sin(t), 3\rangle\, dt$
- [ ] $\iint_S \text{curl}(\mathbf{F}) \cdot \langle 0, 0, 1\rangle\, dS$
- [x] $\iint_D \text{curl}(\mathbf{F}) \cdot \langle 0, 0, 1\rangle\, dS$, where $D$ is the disc $x^2 + y^2 \leq 1$ with $z = 3$
- [ ] $\iint_D \text{curl}(\mathbf{F}) \cdot \langle 0, 0, 1\rangle\, dS$, where $D$ is the disc $x^2 + y^2 \leq 2$ with $z = 0$
- [ ] $\iint_D \text{curl}(\mathbf{F}) \cdot \langle 0, 0, -1\rangle\, dS$, where $D$ is the disc $x^2 + y^2 \leq 1$ with $z = 3$

(c) Use one of the integrals you chose in part (b) to compute the flux of curl($\mathbf{F}$) through $S$. **(4 points)**

$$\iint_S \text{curl}\vec{F}\, d\vec{S} = \iint_D \langle 2yz, 1, 3\rangle \cdot \langle 0,0,1\rangle\, dS = \iint_D 3\, dS$$

$$= 3 \cdot \text{Area}(D) = 3\pi$$

Flux = $\boxed{3\pi}$

**Question 7.**

Let $\mathbf{F} = \langle P, Q \rangle$ where

$P(x,y) = \dfrac{y}{x^2+y^2}$ and $Q(x,y) = 3 + \dfrac{-x}{x^2+y^2}$.

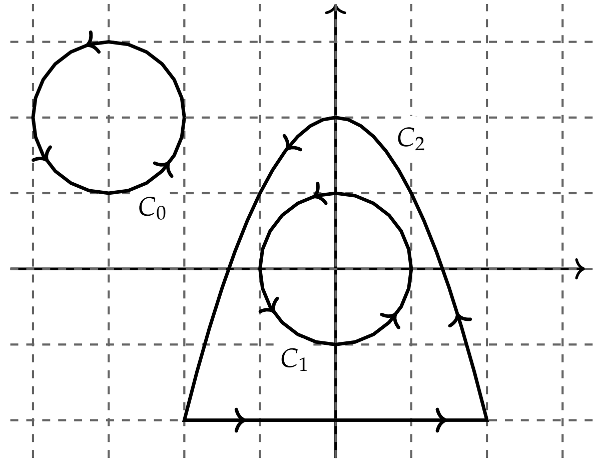
These satisfy

$$P_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$P_y = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q_x = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$Q_y = \frac{2xy}{(x^2+y^2)^2}$$




(a) Let $C_0$ be the circle of radius 1 centered at $(-3, 2)$ and oriented counterclockwise. Compute $\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

| $-3\pi$ | $-2\pi$ | $3-\pi$ | $0$ | $3+\pi$ | $2\pi$ | $3\pi$ |
|---|---|---|---|---|---|---|
| ○ | ○ | ○ | ⊗ | ○ | ○ | ○ |

(b) Let $C_1$ be the circle of radius 1 centered at $(0, 0)$ and oriented counterclockwise. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.
Mark your answer. **(2 points)**

| $-3\pi$ | $-2\pi$ | $3-\pi$ | $0$ | $3+\pi$ | $2\pi$ | $3\pi$ |
|---|---|---|---|---|---|---|
| ○ | ⊗ | ○ | ○ | ○ | ○ | ○ |

(c) Let $C_2$ be the closed curve consisting of the line segment from $(-2, -2)$ to $(2, -2)$ and the graph of $y = 2 - x^2$, oriented as displayed.
Which is the correct answer? **(1 point)**

| $\left\lvert \int_{C_1} \mathbf{F}\cdot d\mathbf{r} \right\rvert < \left\lvert \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \right\rvert$ | $-\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ | $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ | $\left\lvert \int_{C_1} \mathbf{F}\cdot d\mathbf{r} \right\rvert > \left\lvert \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \right\rvert$ |
|---|---|---|---|
| ○ | ○ | ⊗ | ○ |

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**Scratch Space**

parameterize $C_1$

$\vec{r} = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}\,' = \langle -\sin t, \cos t \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{\sin t}{\cos^2 t + \sin^2 t}, 3 - \frac{\cos t}{\cos^2 t + \sin^2 t} \right\rangle \cdot \langle -\sin t, \cos t \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t + 3\cos t - \cos^2 t \, dt = \int_0^{2\pi} -1 + 3\cos t \, dt = -2\pi$$