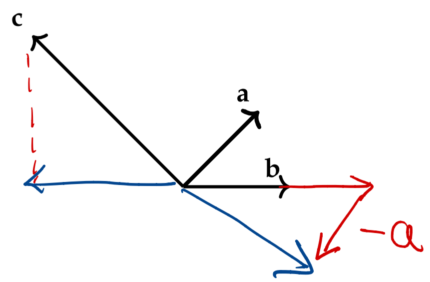
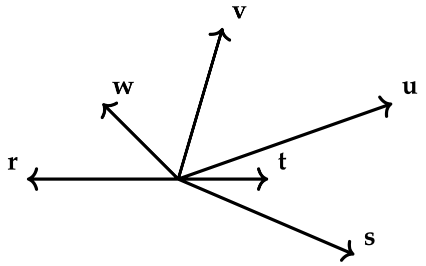
Question 1 **(4 points)** Consider the following vectors in the xy-plane. The vectors $\mathbf{a}$, $\mathbf{b}$ are unit vectors and $|\mathbf{c}| = 2$.





(a) Which vector represents $2\mathbf{b} - \mathbf{a}$? Mark your answer.

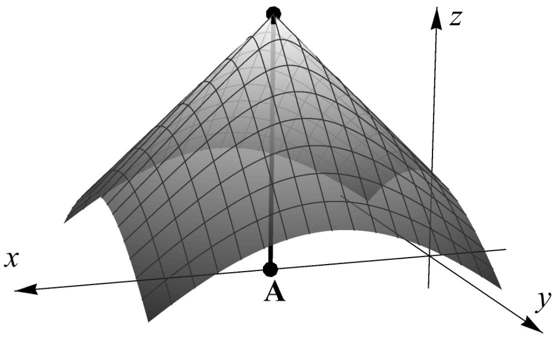
r	**s**	**t**	**0**	**u**	**v**	**w**
○	●	○	○	○	○	○

(b) Which vector represents the projection of $\mathbf{c}$ onto the vector $\mathbf{b}$? Mark your answer.

$\frac{1}{2}\mathbf{s}$	$\mathbf{t}$	$2\mathbf{t}$	$\mathbf{0}$	$\frac{1}{2}\mathbf{u}$	$\mathbf{v}$	$2\mathbf{v}$	$\mathbf{r}$
○	○	○	○	○	○	○	●

Question 2

(2 points) Consider the function $g(x, y)$ whose graph is shown at right. Let A and B be the depicted points in the (x, y)-plane. Mark the answer that is most consistent with the picture.



At the point A, the function g is:

○ differentiable

● continuous

○ both

○ neither

Scratch Space

Question 3 **(4 points)** Consider the function $f(x, y)$ defined by

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0, 0)$? Choose the best answer.

☒ $f(x, y)$ is **not** continuous at $(0, 0)$.

☐ $f(x, y)$ is continuous at $(0, 0)$.

Select the reason which best supports your claim.

☐ The limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = 1$, which can be seen by converting to polar coordinates.

☐ The limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = 1$, which can be seen for example by checking on the line $y = x$.

☐ The limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist because the limits along the curves $y = x^2$ and $y = 0$ are different.

☒ The limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist because the limits along the lines $y = x$ and $y = 0$ are different.

☐ The limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist because the limits along the lines $x = 0$ and $y = 0$ are different.

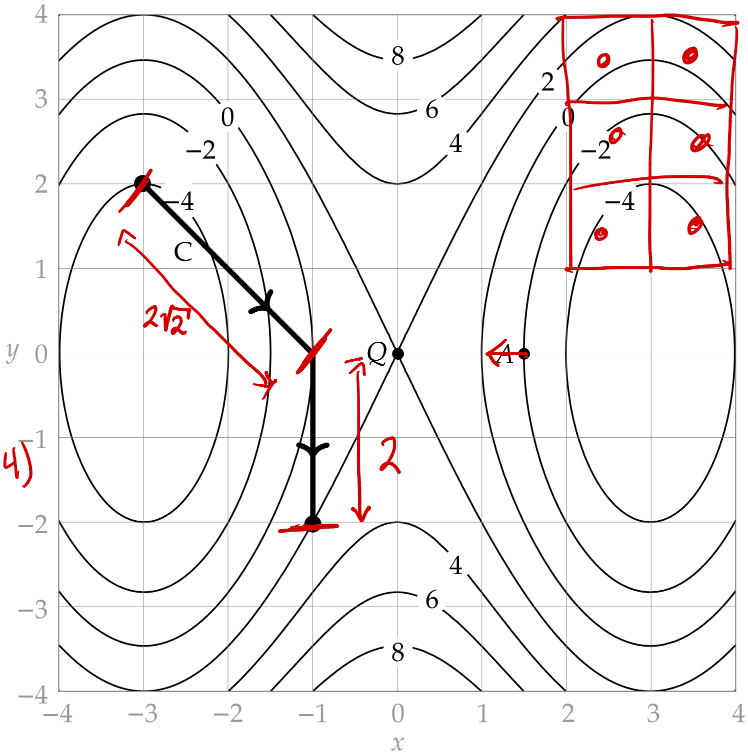
Scratch Space

$$\lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

$$\lim_{(x,x)\to(0,0)} \frac{2x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1$$

Question 4 **(12 points)**

The contour plot of a differentiable function f is shown below. Additionally, a curve C from $(-3,2)$ to $(-1,-2)$ is drawn below. For each part, circle the best answer.



(a) Estimate $\int_1^4 \int_2^4 f(x,y)\,dx\,dy$:

−18	−10	−2	0	2	10	18
○	●	○	○	○	○	○

(b) Estimate $\int_C f\,ds$:

−17	−6.5	−0.6	0	0.6	6.5	17
○	●	○	○	○	○	○

(c) Evaluate $\int_C \nabla f \cdot d\mathbf{r}$: $= f(-1,-2) - f(-3,2) = 2-(-4)$

−10	−6	−2	0	2	6	10
○	○	○	○	○	●	○

(d) Evaluate $\int_C \operatorname{curl}(\nabla f) \cdot d\mathbf{r}$:

−16	−8	−4	0	4	8	16
○	○	○	●	○	○	○

(e) Estimate $\nabla f(A)$:

$\langle -4,0\rangle$	$\langle -2,0\rangle$	$\langle -1,-2\sqrt{2}\rangle$	$\langle 0,-1\rangle$	$\mathbf{0}$	$\langle 0,1\rangle$	$\langle 1,2\sqrt{2}\rangle$	$\langle 2,0\rangle$	$\langle 4,0\rangle$
●	○	○	○	○	○	○	○	○

$f_y = 0$

$f_x \approx \dfrac{-4-0}{2-1} = -4$

(f) The point Q is:

a local minimum	a saddle point	a local maximum	not a critical point
○	●	○	○

Scratch Space

(a) $\int_1^4 \int_2^4 f(x,y)\,dx\,dy \approx \left(-4 + -4 + -2 + -2 + 1 + 1 \right) = -10$

(b) $\int_C \nabla f \cdot d\vec{r} = \text{length}(C)\ \text{Average}(f) \approx \left[2\sqrt{2} \cdot (-3) + 2 \cdot (1) \right] \approx 3(-3) + 2 = -7$

Question 5 (4 points) The function $f(x,y)$ describes the temperature (°C) in a given region R in the plane, so that $f(x,y)$ is the temperature at position (x,y). A few values of f together with its rates of change are given in the following table. Assuming that f is differentiable, use this data and linear approximation to estimate the temperature at $(1.2, 2.4)$.

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{yy}(x,y)$	$f_{xy}(x,y)$
$(1,2)$	2	−3	1	6	7	8
$(0.2,0.4)$	−1	−2	−4	1	−5	−6

$$f(1.2,2.4) \approx f(1,2) + f_x(1,2)(1.2-1) + f_y(1,2)(2.4-2)$$

$$= 2 + (-3)(0.2) + (1)(0.4)$$

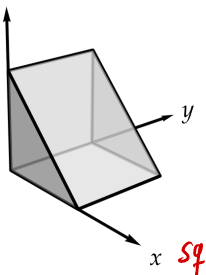
$$= 2 - 0.6 + 0.4$$

$f(1.2, 2.4) \approx$ 1.8

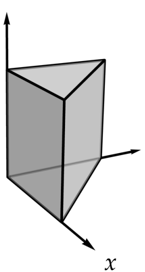
Question 6 (4 points)

Label each integral with its corresponding region of integration. Write your answer in the box next to the integral.

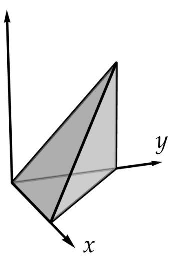
(A)



(B)



(C)

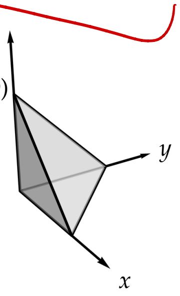


F $\int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} f(x,y,z)\, dx\, dy\, dz$

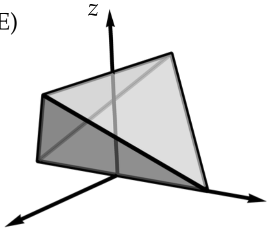
square in xz-plane

B $\underbrace{\int_0^1 \int_0^1} \int_0^{1-x} g(x,y,z)\, dy\, dx\, dz$

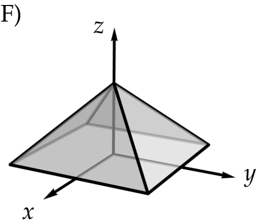
(D)



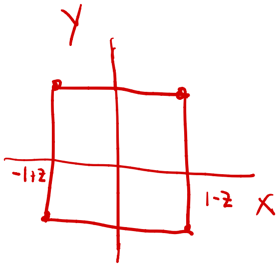
(E)

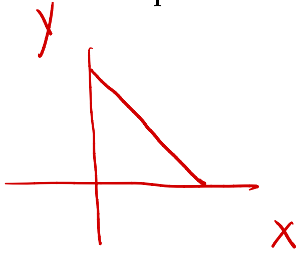


(F)



Scratch Space





Question 7 **(5 points)**

Consider the differentiable function $f(x, y)$. The table lists values of partial derivatives of f at several points. For each of the listed points below determine whether it is a local minimum, local maximum, saddle point, or none of these. Mark your answer below.

(x, y)	$f_x(x, y)$	$f_y(x, y)$	$f_{xx}(x, y)$	$f_{yy}(x, y)$	$f_{xy}(x, y)$
$(0, 0)$	-3	0	0	0	0
$(-1, 0)$	0	0	0	0	-6
$(0, 1)$	0	0	6	6	0

$(0, 0)$ is: ● not a critical point ○ a local minimum ○ a local maximum ○ a saddle point

$(-1, 0)$ is: ○ not a critical point ○ a local minimum ○ a local maximum ● a saddle point

$(0, 1)$ is: ○ not a critical point ● a local minimum ○ a local maximum ○ a saddle point

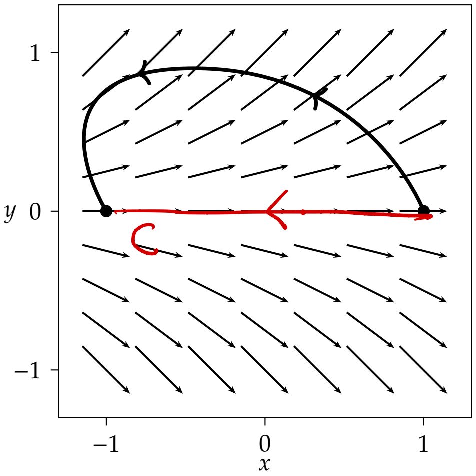
Question 8 **(2 points)**

A conservative force field **F** is shown on the right. For scale, $\mathbf{F}(0, 0) = \langle \frac{\sqrt{2}}{5}, 0 \rangle$. Estimate the work done by **F** to move a particle from $(1, 0)$ to $(-1, 0)$ along the indicated path.

-2	$-\frac{2\sqrt{2}}{5}$	$-\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{2\sqrt{2}}{5}$	2
○	●	○	○	○	○	○

FTLI ⇒ can compute line integral along C

$\approx \left(\frac{-\sqrt{2}}{5}\right)(2) = -2\sqrt{2}/5$

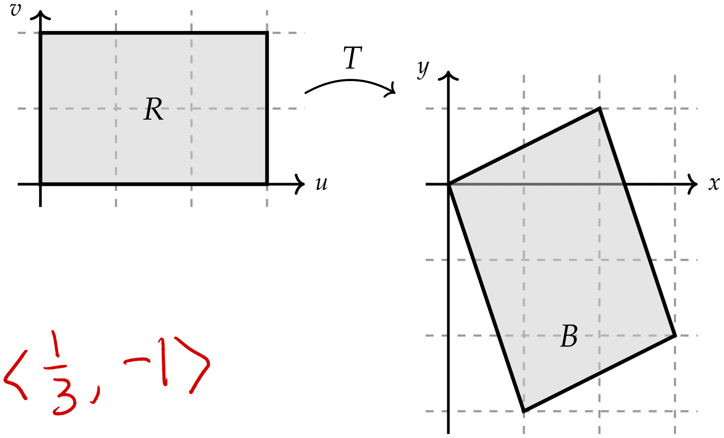


Scratch Space

$D(-1,0) = f_{xx}(-1,0)\, f_{yy}(-1,0) - f_{xy}(-1,0)^2 = 0 - (-6)^2 = -36 < 0$

$D(0,1) = f_{xx}(0,1)\, f_{yy}(0,1) - f_{xy}(0,1)^2 = (6)(6) - 0 = 36 > 0 ;\ f_{xx}(0,1) > 0$

Question 9 (4 points)
Let B be the parallelogram bounded by the lines $y = -3x$, $y = -3x + 7$, $x = 2y$, and $x = 2y + 7$, and let R be the rectangle in the (u, v)-plane with vertices $(0, 0)$, $(3, 0)$, $(3, 2)$, and $(0, 2)$.
Find a linear transformation T that takes the rectangle R to the parallelogram B.



$T(1,0) = \frac{1}{3}\langle 1,-3\rangle = \langle \frac{1}{3}, -1\rangle$

$T(0,1) = \frac{1}{2}\langle 2,1\rangle = \langle 1, \frac{1}{2}\rangle$

$T(u,v) = \boxed{\left\langle \frac{u}{3} + v \quad , -u + \frac{v}{2} \right\rangle}$

Scratch Space

Question 10 (4 points) Let C be the curve parameterized by $\mathbf{r}(t) = -2t^2\mathbf{i} + (t-1)\mathbf{j} + t^3\mathbf{k}$. Find a vector equation for the tangent line to C at $(-2, 0, 1)$.

$(-2,0,1)$ corresponds to $t=1$

$\vec{r}\,'(t) = \langle -4t, 1, 3t^2 \rangle \rightsquigarrow \vec{r}\,'(1) = \langle -4, 1, 3 \rangle$ ← direction of tangent line

Tangent line: $\boxed{\langle -2-4t \;,\; t \;,\; 1+3t \rangle}$

Question 11 (2 points)
Consider the four electric charges placed as follows.
Charge Q_1 with value -4 is placed at (0, −5, 0),

Charge Q_2 with value 1 is placed at (3, 0, 0),

Charge Q_3 with value 5 is placed at (0, 0, 0), and

Charge Q_4 with value 3 is placed at (0, 0, −1).

Let $\mathbf{E}$ be the resulting electric field. The flux of $\mathbf{E}$ across the sphere $x^2 + y^2 + z^2 = 36$ is equal to $\frac{5}{\epsilon_0}$.

Let S be the region $\frac{x^2}{4} + y^2 + \frac{z^2}{9} \le 1$. Determine the flux of $\mathbf{E}$ across ∂S. Mark your answer.

-2π	$-\pi$	$-\frac{8}{\epsilon_0}$	$-\frac{5}{\epsilon_0}$	$-\frac{1}{\epsilon_0}$	0	$\frac{1}{\epsilon_0}$	$\frac{5}{\epsilon_0}$	$\frac{8}{\epsilon_0}$	π	2π
○	○	○	○	○	○	○	○	●	○	○

Scratch Space

Which charges are inside of S?

$Q_1 \leftarrow$ not in S, $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 0 + (-5)^2 + 0 = 25 > 1$

$Q_2 \leftarrow$ not in S, $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = \frac{9}{4} + 0 + 0 = \frac{9}{4} > 1$

$Q_3 \leftarrow$ in S

$Q_4 \leftarrow$ in S

Gauss's Law $\Rightarrow$ flux across ∂S is $\frac{Q_3}{\epsilon_0} + \frac{Q_4}{\epsilon_0} = \frac{5+3}{\epsilon_0}$

Question 12 **(4 points)** Assume $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a vector field defined on the shaded region D depicted in the diagram, and assume that P and Q have continuous first partial derivatives on D. The region D is defined as

$$D = \left\{(x, y) | x^2 + \left(\frac{5y}{4} - \sqrt{|x|}\right)^2 < 1 \text{ and } \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{4}\right)^2 > \frac{1}{20}\right\}$$

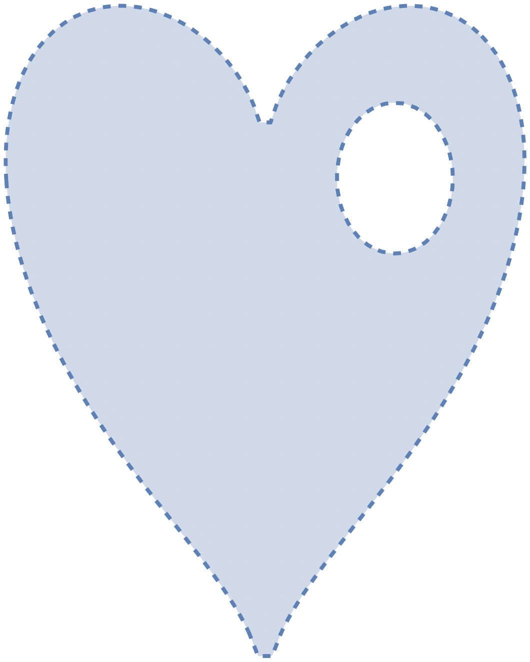
(a) **(2 points)** Which of the following statements about D is correct? (Mark all that apply.)

☒ D is bounded

☒ D is connected

☐ D is simply connected

☒ D is open



(b) **(2 points)** Assume $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Which of the following statements about $\mathbf{F}$ on the region D are correct? (Mark exactly one option.)

○ $\mathbf{F}$ cannot be conservative

○ $\mathbf{F}$ must be conservative

● $\mathbf{F}$ may or may not be conservative

Scratch Space

Question 13 (4 points)
Let V be the solid lying above the plane $z = -1$, below the surface $z = x^2 + y^2$, and inside the cylinder $x^2 + y^2 = 4$. In cylindrical coordinates, the mass of V is computed by an integral of the form

$$\int_f^e \int_d^c \int_b^a g(r, \theta, z)\, d?\, d?\, d?$$

(a) Mark the integral below with the correct bounds of integration. Pay attention to the given orders of integration.

● $\int_0^{2\pi} \int_0^{2} \int_{-1}^{r^2} g(r, \theta, z)\, dz\, dr\, d\theta$

○ $\int_0^{2\pi} \int_{\sqrt{z}}^{2} \int_{-1}^{4} g(r, \theta, z)\, dz\, dr\, d\theta$

○ $\int_0^{2\pi} \int_0^{4} \int_{-1}^{r^2} g(r, \theta, z)\, dz\, dr\, d\theta$

○ $\int_0^{2\pi} \int_{\sqrt{z}}^{2} \int_{-1}^{r^2} g(r, \theta, z)\, dz\, dr\, d\theta$

○ $\int_0^{2\pi} \int_{-1}^{4} \int_{\sqrt{z}}^{2} g(r, \theta, z)\, dr\, dz\, d\theta$

○ $\int_0^{2\pi} \int_{\sqrt{z}}^{2} \int_{-1}^{4} g(r, \theta, z)\, dr\, dz\, d\theta$

○ $\int_0^{2\pi} \int_{-1}^{4} \int_{2}^{\sqrt{z}} g(r, \theta, z)\, dr\, dz\, d\theta$

○ $\int_0^{2\pi} \int_0^{2} \int_0^{1} g(r, \theta, z)\, dr\, dz\, d\theta$

(b) Mark the integral with correct integrand if the mass density is $\rho(x, y, z) = 2zx$.

○ $\int_f^e \int_d^c \int_b^a 2r^2 \cos\theta \sin\theta\, d\theta\, dr\, dz$

○ $\int_f^e \int_d^c \int_b^a 2r^3 \cos\theta \sin\theta\, d\theta\, dr\, dz$

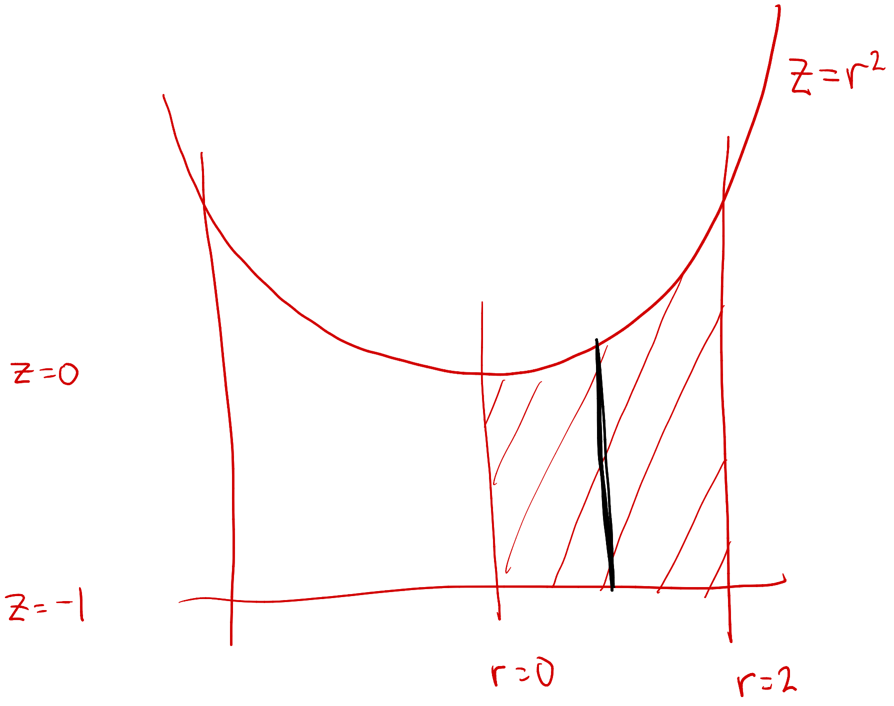
○ $\int_f^e \int_d^c \int_b^a 2zr^3 \cos\theta \sin\theta\, d\theta\, dr\, dz$

○ $\int_f^e \int_d^c \int_b^a 2zr \cos\theta\, d\theta\, dr\, dz$

● $\int_f^e \int_d^c \int_b^a 2zr^2 \cos\theta\, d\theta\, dr\, dz$

○ $\int_f^e \int_d^c \int_b^a 2zr^3 \cos\theta \sin\theta\, d\theta\, dr\, dz$

Scratch Space



$\rho(r,\theta,z) = 2z r\cos\theta$

$dV = r\, dr\, d\theta\, dz$

Question 14 **(9 points)** Let $\mathbf{F} = \langle z - xy, -z^2, 4x - yz \rangle$.

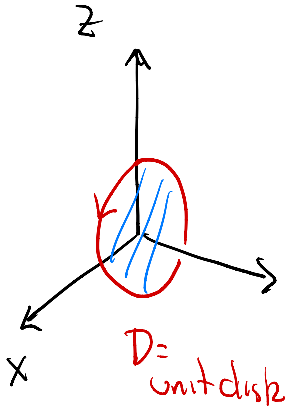
(a) Compute curl($\mathbf{F}$). Mark your answer.

$\langle -z, -3, -x \rangle$	$\langle -z, 3, -x \rangle$	$\langle -y, 3, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle z, 3, x \rangle$	$\langle z, -3, x \rangle$	$\langle 0, 3, -x \rangle$
○	○	○	○	○	●	○

(b) Compute div($\mathbf{F}$).

$-2y$	$-x - z - 1$	$1 - x - z$	0	$x + y + z$	$xy + yz$	$2y$
●	○	○	○	○	○	○

(c) Let C be the curve $x^2 + z^2 = 1$ oriented counterclockwise in the xz-plane, as viewed from the positive y-axis. Use Stokes's Theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.



D= unit disk in xz-plane

Stokes: $\int_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}\vec{F} \cdot \langle 0,1,0 \rangle \, dS$

$= \iint_D \langle z, -3, x \rangle \cdot \langle 0,1,0 \rangle \, dS$

$= -3(\text{area}(D)) =$

$\int_C \mathbf{F} \cdot d\mathbf{r} = \boxed{-3\pi}$

Scratch Space

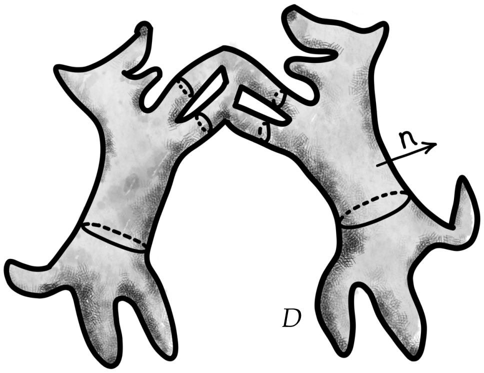
$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ z - xy & -z^2 & 4x - yz \end{vmatrix} = \langle -z + 2z, -(4-1), 0 - -x \rangle$$

$$= \langle z, -3, x \rangle$$

$\text{div}\,\vec{F} = -y + 0 - y \quad = -2y$

Question 15 (8 points)

Let S and D be the depicted surfaces, oriented outwards; their unit normals are also pictured. The boundary of S is the unit circle in the xy-plane and D has no boundary.

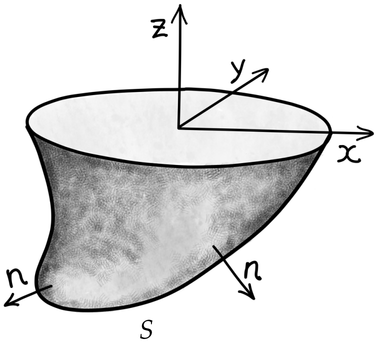


(a) Let $\mathbf{F} = \langle z - x,\ x,\ y - z\rangle$. The integral $\iint_D \mathbf{F}\cdot d\mathbf{S}$ is:

positive	zero	negative
○	○	●

(b) The flux of curl($\mathbf{F}$) = $\langle 1, 1, 1\rangle$ across D is:

positive	zero	negative
○	●	○



(c) Let C be the unit circle in the xy-plane, oriented counterclockwise as viewed from above. The integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is:

-2π	$-\pi$	-2	0	2	π	2π
○	○	○	○	○	●	○

(d) Let $\mathbf{G} = \langle z,\ x + 2z,\ 3\rangle$. The flux of G through S is:

-3π	-2π	$-\pi$	-3	-2	0	2	3	π	2π	3π
●	○	○	○	○	○	○	○	○	○	○

Scratch Space

(a) $\operatorname{div}\vec{F} = -1+0-1 = -2$, D is boundary of solid region B

divergence theorem $\Rightarrow \iint_D \vec{F}\cdot d\vec{S} = \iiint_B \operatorname{div}\vec{F}\, dV = \iiint_B -2\, dV = -2\cdot \text{Volume}(B)$

(c) Note: ∂S has opposite orientation as C.

Stokes $\Rightarrow \int_C \vec{F}\cdot d\vec{r} = -\int_{\partial S} \vec{F}\cdot d\vec{r} = -\iint_T \text{curl}\vec{F}\cdot d\vec{S} = -\iint_T \langle 1,1,1\rangle\cdot\langle 0,0,-1\rangle\, dS$

unit disk oriented downwards

$= \pi$

(d) $\operatorname{div}\vec{G} = 0$; divergence theorem $\Rightarrow \iint_S \vec{G}\cdot d\vec{S} = \iint_T \vec{G}\cdot d\vec{S} = \iint_T \langle z, x+2z, 3\rangle\cdot\langle 0,0,-1\rangle$

unit disk oriented downwards

$= -3\pi$

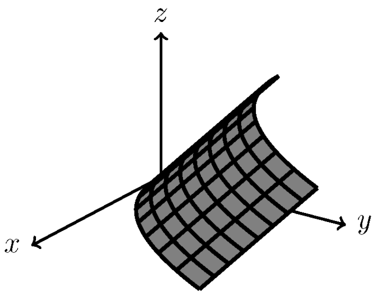
Question 16 **(8 points)**

Consider the surface S parameterized by $\mathbf{r}(u,v) = \langle u, u^2+v, v\rangle$ with domain $D = \{0 \le u \le 1,\ 0 \le v \le 1\}$.

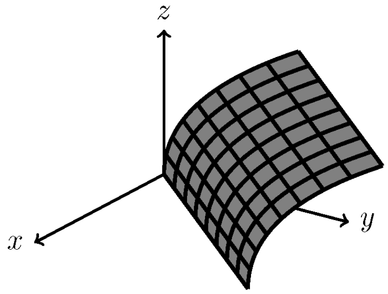
(a) Mark the picture below which corresponds to S.

$\vec r(0,v) = \langle 0,v,v\rangle$, $\vec r(u,0) = \langle u,u^2,0\rangle$

$y=z$ $\quad$ $y=x^2$



●



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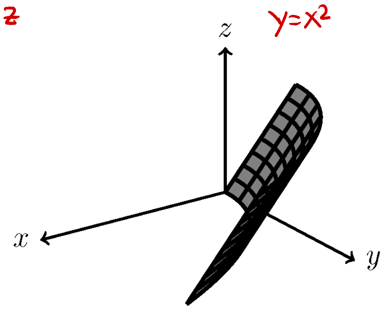

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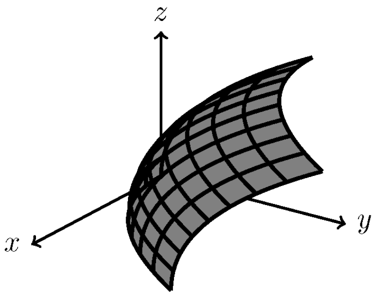

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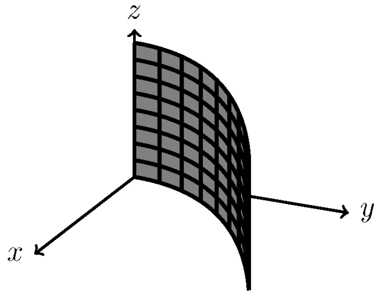

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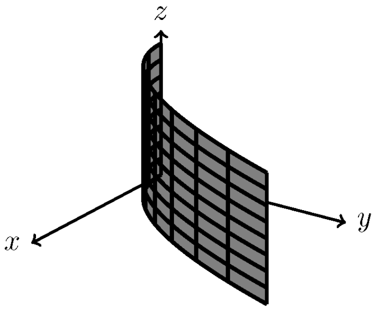

○

(b) Which one of the following integrals computes the surface area of S?

$\iint_D |\vec r_u \times \vec r_v|\, du\,dv$

○ $\int_0^1 \int_0^1 2u\, du\, dv$

● $\int_0^1 \int_0^1 \sqrt{4u^2+2}\, du\, dv$

○ $\int_0^1 \int_0^1 1\, du\, dv$

○ $\int_0^1 \int_0^1 \sqrt{u^2+u^4+2uv+2v^2}\, du\, dv$

○ $\int_0^1 \int_0^1 \sqrt{2}\, du\, dv$

● $\int_0^1 \int_0^1 \sqrt{4u^2+2}\, du\, dv$

(c) Orient S in the direction of the positive y-axis, that is, with a unit normal vector $\mathbf{n}$ whose second component is positive. Which one of the following integrals computes the flux of $\mathbf{F} = \langle x, 3, z^2\rangle$ across S? Mark the correct answer.

○ $\int_0^1 \int_0^1 2u^2+v^2-3\, du\, dv$

○ $\int_0^1 \int_0^1 3\, du\, dv$

○ $\int_0^1 \int_0^1 -3\, du\, dv$

○ $\int_{\partial S} \langle x, 3, z^2\rangle \cdot d\mathbf{r}$

● $\int_0^1 \int_0^1 3-2u^2-v^2\, du\, dv = \int_0^1\int_0^1 \langle u,3,v^2\rangle \cdot \vec r_u \times \vec r_v\, dudv$

○ $-\int_{\partial S} \langle x, 3, z^2\rangle \cdot d\mathbf{r}$

$= \int_0^1\int_0^1 \langle u,3,v^2\rangle \cdot \langle -2u,1,-1\rangle\, dudv$

$= \int_0^1\int_0^1 -2u^2+3-v^2\, dudv$

(d) For the vector field $\mathbf{F} = \langle x, 3, z^2\rangle$ from part (c), what is the sign of $\iint_S \operatorname{div}\mathbf{F}\, dS$? Mark your answer.

$\iint_S \operatorname{div}\mathbf{F}\, dS$ is: negative ○ zero ○ positive ●

$\operatorname{div}\vec F = 1+2z$, positive on S

$\vec r_u \times \vec r_v = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 2u & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 2u, -1, 1\rangle$, $|\vec r_u \times \vec r_v| = \sqrt{4u^2+2}$

points in negative y-direction