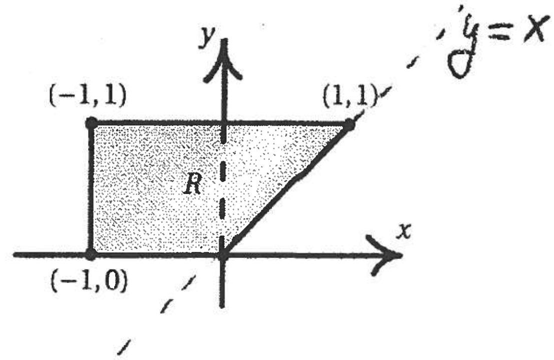
1. For the region R at right, evaluate $\iint_R 2x\, dA$. **(4 points)**



$$\iint_R 2x\, dA = \int_0^1 \int_{-1}^{y} 2x\, dx\, dy$$

$$= \int_0^1 x^2 \Big|_{x=-1}^{y} dy = \int_0^1 y^2 - 1\, dy$$

$$= \frac{y^3}{3} - y \Big|_{y=0}^{1} = \frac{1}{3} - 1 = -2/3$$

$$\boxed{\iint_R 2x\, dA = -2/3}$$

2. Consider the ellipse C given by $x^2 - xy + y^2 = 1$. Find all the points on C which are closest to the origin. **(6 points)**

Want to minimize $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x^2 - xy + y^2 = 1$. Use Lagrange Multiplers.

$$\nabla f = (2x, 2y) = \lambda \nabla g = \lambda(2x - y, 2y - x)$$

$$\Rightarrow 2x = \lambda(2x - y) \text{ and } 2y = \lambda(2y - x)$$

$$\Rightarrow \frac{1}{\lambda} = 1 - \frac{y}{2x} = 1 - \frac{x}{2y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2 = y^2$$

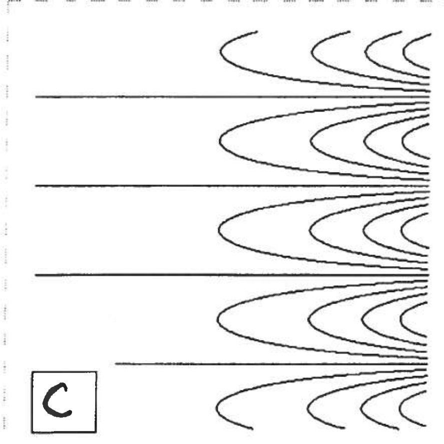
Case $x = y$: $g = x^2 = 1 \Rightarrow$ Two crit pts: $(1,1)$, $(-1,-1)$ where $f = 2$

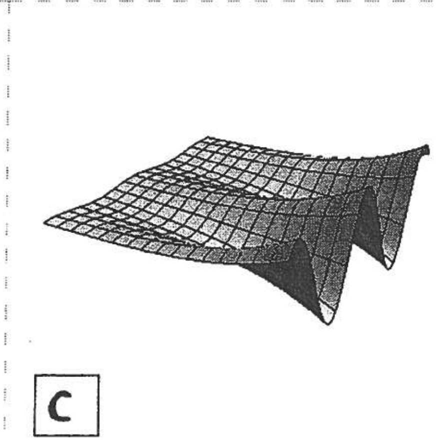
Case $x = -y$: $g = 3x^2 = 1 \Rightarrow$ Two crit pts: $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$, $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ where $f = 2/3$, so these are the closest pts

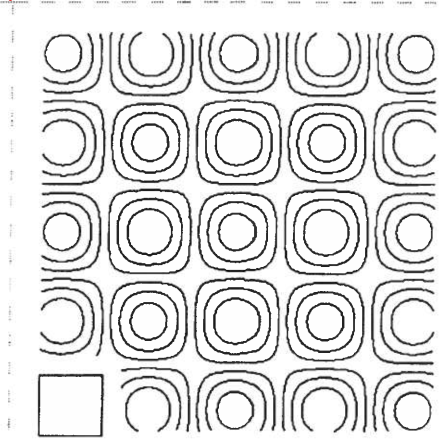
Closest points: $\left(1/\sqrt{3}, -1/\sqrt{3}\right), \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

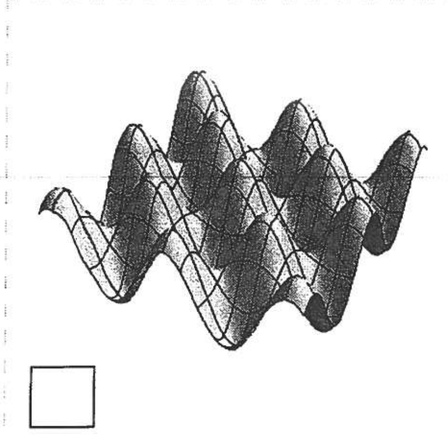
3. For each function: (a) xy (b) $\cos\left(\sqrt{x^2+y^2}\right)$ (c) $e^x \cos y$

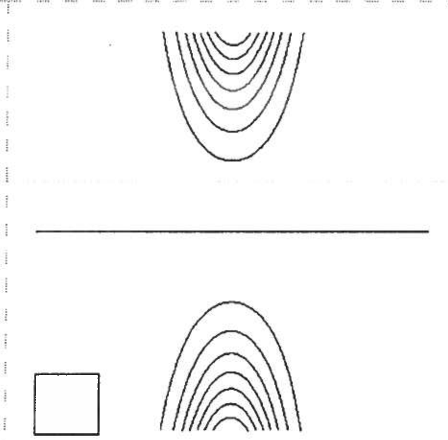
label its graph and its level set diagram from among the options below. Here each level set diagram consists of level sets $\{f(x,y)=c_i\}$ drawn for evenly spaced c_i. **(9 points)** explanations on next page

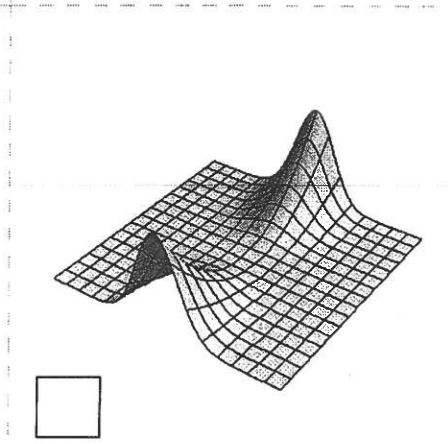


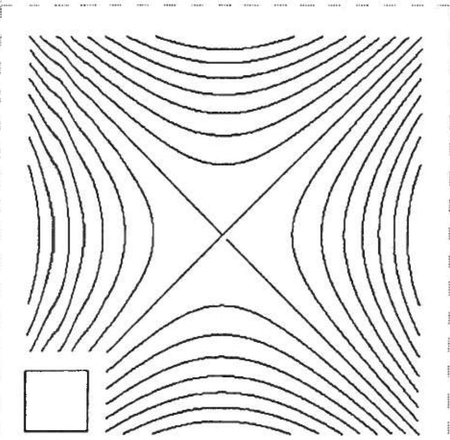


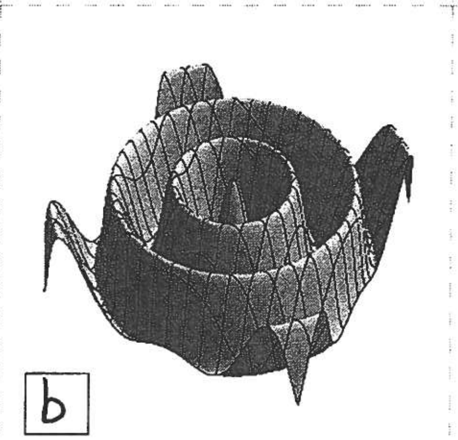


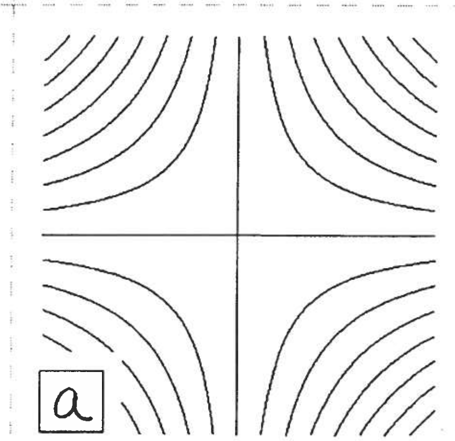


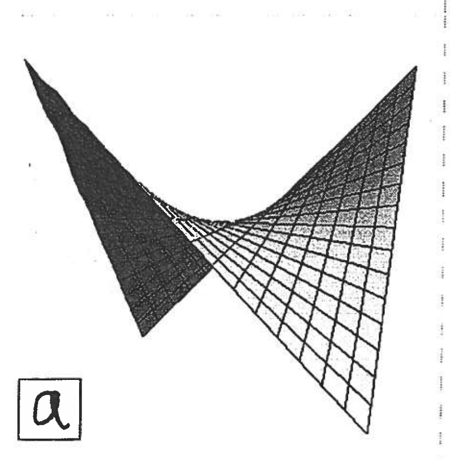


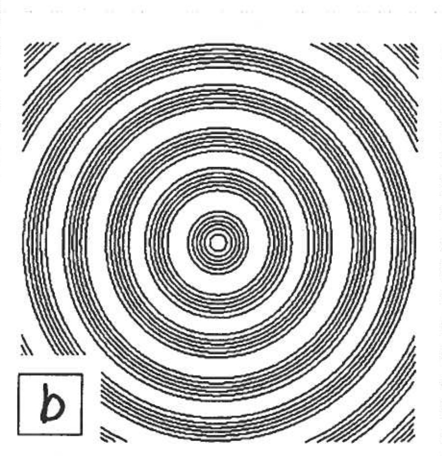


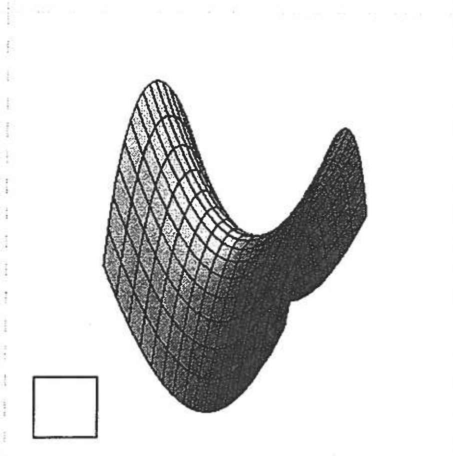












Problem 3

(a) Note that the function xy is constantly 0 along the lines $x = 0$ and $y = 0$, so the contour plot should have those lines. The third row on the right is the only one which has a vertical line as a contour curve (this is $x = 0$).

In all the graphs, the grid curves are the intersections of the graphs with the planes $x =$ constant and $y =$ constant. If you fix a constant value of x, say $x = 2$ for example, the line $z = 2y$ is one of the grid curves on the graph of xy. The only graph whose grid curves are all straight lines is the bottom row on the left. (It may not look like they are straight, since the surface itself curves, but if you hold a pencil or the edge of a paper to each individual grid curve, you will see they are all straight.)

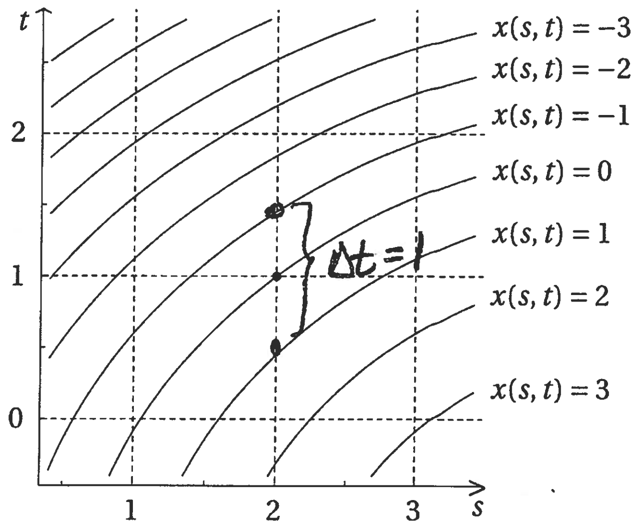
(b) Any time you see $\sqrt{x^2+y^2}$, looking at polar coordinates may be helpful. For (b), the function in polar coordinates is $\cos(r)$. Any function that has only r but no θ in polar coordinates is rotationally symmetric around the origin. The only graph which has this rotational symmetry is the third row, middle.

For a similar reason, any function depending on r but not θ must have a contour plot containing only concentric circles around $(0, 0)$. Bottom row, middle is the only contour plot like this.

(c) e^x decays to 0 as $x \to -\infty$, so the graph must flatten out as x decreases. As x increases, however, e^x becomes large, so the graph should look more like a big cosine wave. Top row, middle is the only graph fitting this description.

If you fix a value of x, the function is a multiple of $\cos(y)$, so the contour plot should start to repeat as you move up and down. But if you fix y, the function decays to 0 on the left and gets larger quickly on the right, so the contour plot should have contour closer together on the right and further apart on the left. Top row, left is the only contour plot like this.

4. Let $x(s,t)$ be the function whose contour plot is shown at right.



(a) Estimate $\frac{\partial x}{\partial t}(2,1)$ and circle the closest number below. **(2 points)**

$$\approx \frac{\Delta x}{\Delta t} = \frac{-2}{1} = -2$$

$\frac{\partial x}{\partial t}(2,1)$: $\quad -6 \quad -4 \quad \boxed{-2} \quad 0 \quad 2 \quad 4 \quad 6$

(b) Let $g(x,y)$ be a function with the table of values and partial derivatives shown below and let $y(s,t) = s + t$. For $G(s,t) = g(x(s,t), y(s,t))$, compute $\frac{\partial G}{\partial t}(2,1)$. **(4 points)**

(x,y)	$g(x,y)$	$\frac{\partial g}{\partial x}$	$\frac{\partial g}{\partial y}$
(0,3)	0	3	6
(0,1)	2	-3	-1
(2,1)	3	4	7
(3,3)	1	3	5

$x(2,1) = 0$

$y(2,1) = 3$

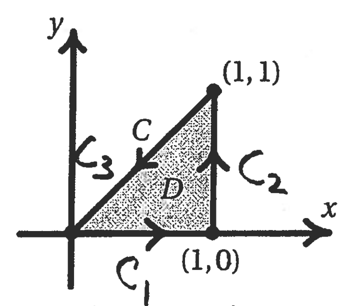
$$\frac{\partial G}{\partial t}(2,1) = \frac{\partial g}{\partial x}(x(2,1), y(2,1))\frac{\partial x}{\partial t}(2,1) + \frac{\partial g}{\partial y}(x(2,1), y(2,1))\frac{\partial y}{\partial t}(2,1)$$

$= 1$

$$= 3 \cdot (-2) + 6 \cdot 1 = 0$$

$\frac{\partial G}{\partial t}(2,1) = 0$

5. (a) For the curve C at right, directly compute $\int_C y\,dx + 3x\,dy$. **(5 points)**



Bottom: $y = 0$ and $dy = 0$ so $\int_{C_1} = \boxed{0.}$

Right: $dx = 0$ and $x = 1$,

so $\int_{C_2} = \int_0^1 3\,dy = \boxed{3.}$

Diagonal: $\vec{r}(t) = (1-t, 1-t)$ for $0 \le t \le 1$. So $dx = dy = -dt$

and so $\int_{C_3} y\,dx + 3x\,dy = \int_0^1 4(t-1)\,dt$

$= 2t^2 - 4\Big|_0^1 = \boxed{-2}$

$0+3-2$

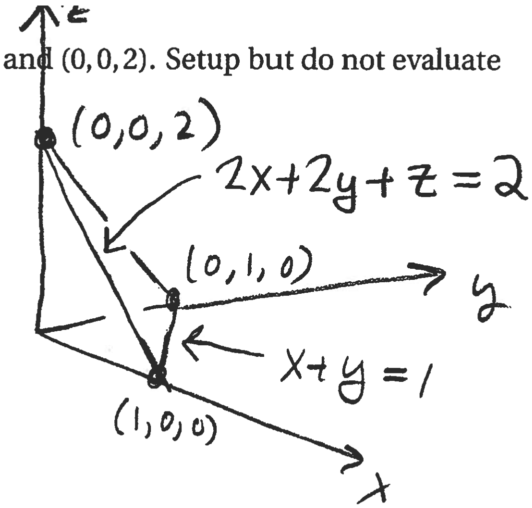
$\int_C y\,dx + 3x\,dy = 1$

(b) Check your answer in (a) using Green's Theorem. **(2 points)**

$$\int_C \underset{P}{y}\,dx + \underset{Q}{3x}\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \iint_D 2\,dA$$

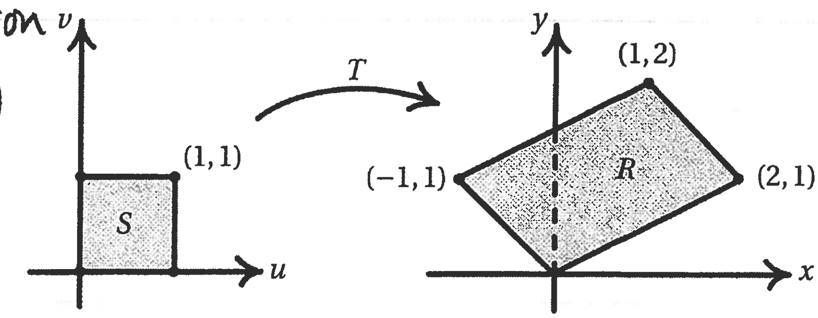
$$= 2\,\text{Area}(D) = \boxed{1}$$ ✓

6. Let E be the tetrahedron in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,2)$. Setup but do not evaluate a triple integral that computes the volume of E. **(5 points)**



$$\int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} 1 \, dz\, dy\, dx$$

7. (a) Let R be the region shown. Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [0,1] \times [0,1]$ to R. **(3 points)**



Use a linear transformation

$T(u,v) = (au+bv, cu+dv)$

Want:

$T(1,0) = (a,c) = (2,1)$

$T(0,1) = (b,d) = (-1,1)$

So $a=2, b=-1, c=1, d=1$

$$\boxed{T(u,v) = (2u - v \ , \ u+v)}$$

(b) Use your change of coordinates to evaluate $\iint_R x\,dA$ via an integral over S. **(5 points)**

Emergency backup transformation: If you can't do (a), pretend you got the answer $T(u,v) = (uv, v)$ and do part (b) anyway.

$$\iint_R x\,dA = \int_0^1\int_0^1 (2u-v)\, 3 \, du\, dv = 3\int_0^1 u^2 - uv \Big|_{u=0}^{1} dv$$

$$= 3\int_0^1 1 - v \, dv$$

$$= 3\left(v - \frac{v^2}{2}\right)\Big|_0^1 = 3/2$$

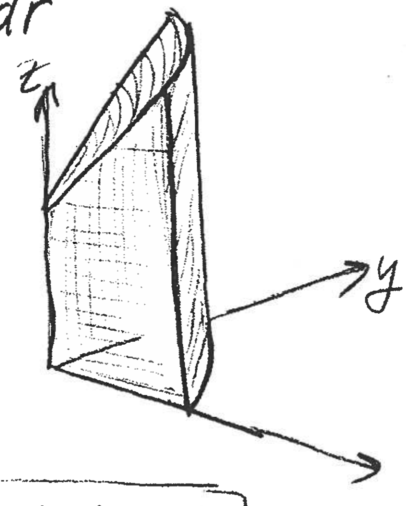
$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$|\det J| = |2+1| = 3$$

$$\boxed{\iint_R x\,dA = 3/2}$$

8. Let R be the portion of the cylinder $x^2 + y^2 \leq 1$ which lies in the octant where $\{x \geq 0, y \geq 0, z \geq 0\}$ and lies below the cone $z = 1 + \sqrt{x^2 + y^2}$. For the density $\rho = 6z$, compute the total mass of R. **(7 points)**

$$\text{Mass} = \iiint_R \rho \, dV = \int_0^1 \int_0^{1+r} \int_0^{\pi/2} 6zr \, d\theta \, dz \, dr$$



$$= \int_0^1 \int_0^{1+r} 3\pi z r \, dz \, dr$$

$$= \int_0^1 \frac{3\pi}{2} z^2 \Big|_{z=0}^{1+r} r \, dr$$

$$= \frac{3\pi}{2} \int_0^1 (r^2 + 2r + 1) r \, dr$$

Cylindrical Coordinates

$$= \frac{3\pi}{2} \left[\frac{r^4}{4} + \frac{2r^3}{3} + \frac{r^2}{2} \right]_{r=0}^{1} = \frac{3\pi}{2}\left(\frac{17}{12}\right) + \frac{17\pi}{8}$$

$$\boxed{\text{Mass} = \frac{17\pi}{8}}$$

9. Let $\mathbf{F}(x, y, z) = \left\langle \frac{x^3}{3}, x^2 \cos(z) + \frac{y^3}{3}, \frac{z^3}{3} \right\rangle$ and let S denote the surface defined by $x^2 + y^2 + z^2 = 1$, equipped with the inward-pointing unit normal vector field $\mathbf{n}$. Compute $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ by any valid method. **(6 points)**

dA

inward pointing

$$\iint_S \vec{F} \cdot \vec{n} \, dA \underset{\text{divergence thm}}{=} -\iiint_{Ball} \operatorname{div} \vec{F} \, dV = -\iiint_{Ball} \underbrace{x^2 + y^2 + z^2}_{\rho^2} \, dV$$

$$= -\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

change to Spherical

$$= -\int_0^{2\pi} \int_0^{\pi} \sin\phi \frac{\rho^5}{5} \Big|_{\rho=0}^{1} d\phi \, d\theta$$

$$= -\frac{1}{5} \int_0^{2\pi} \int_0^{\pi} \sin\phi \, d\phi \, d\theta = +\frac{1}{5} \int_0^{2\pi} \cos\phi \Big|_{\phi=0}^{\phi=\pi} d\theta$$

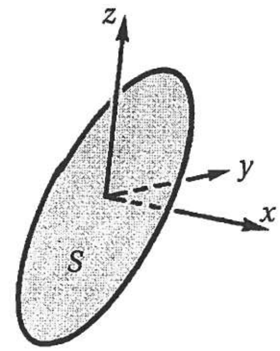
$$= -\frac{2}{5} \int_0^{2\pi} 1 \, d\theta = -\frac{4\pi}{5}$$

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, \cancel{dS}\, dA = -\frac{4\pi}{5}}$$

10. Consider the surface S which is the portion of the plane $z = x + y$ which lies inside the cylinder $x^2 + z^2 = 4$. Give a parameterization $\mathbf{r}: D \to S$ where D is a rectangle in plane with coordinates u and v. **(5 points)**

Idea: Use rotated polar on cylinder:

$x = v\cos u \qquad z = v\sin u$



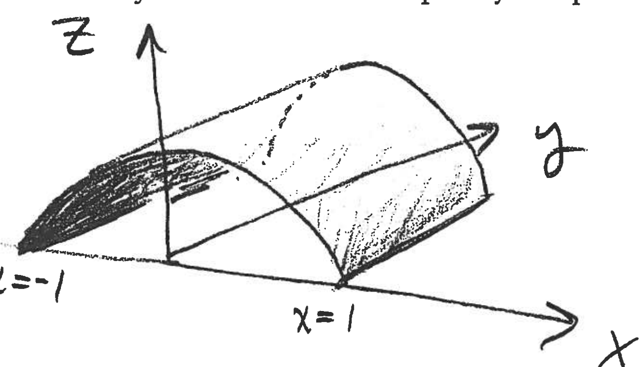
Now solve for y: $y = z - x$

$= v\sin u - v\cos u$

$D = \{\, 0 \le u \le 2\pi$ and $0 \le v \le 2 \,\}$

$\mathbf{r}(u, v) = \langle\, v\cos u \;,\; v(\sin u - \cos u),\; v\sin u \,\rangle$

11. Consider the portion S of the surface $z = 1 - x^2$ where $0 \le y \le 1$ and $z \ge 0$. Completely setup but do not evaluate $\iint_S x^2\, dA$. **(5 points)**



Param: $\vec{r}(u,v) = (u, v, 1-u^2)$

$-1 \le u \le 1$ and $0 \le v \le 1$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = \langle 2u, 0, 1 \rangle$$

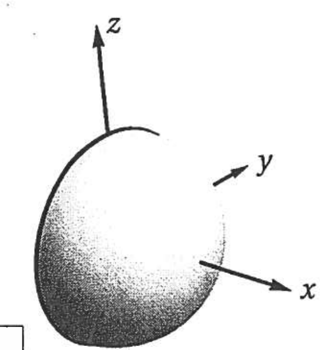
$$\iint_S x^2\, dA = \int_0^1 \int_{-1}^1 u^2 |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$\boxed{= \int_0^1 \int_{-1}^1 u^2 \sqrt{1+4u^2}\, du\, dv.}$$

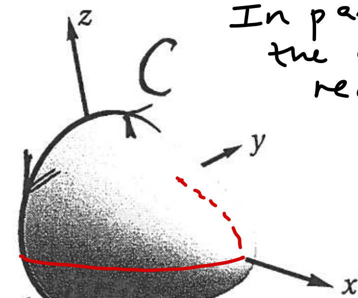
12. Let S be the surface in $\mathbb{R}^3$ parameterized by $\mathbf{r}(u,v) = \langle 2-2v^2,\ v\cos u,\ v\sin u\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$.

(a) Mark the correct picture of S below. **(2 points)**

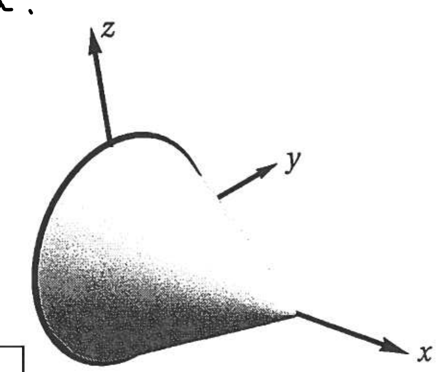
Note that $r(0,v) = (2-2v^2, v, 0)$ is a parabola. In particular the one in red here.



☐



☑



☐

also in cylindrical coordinates about the x-axis, with ρ being radius, satisfies the equation of the paraboloid $x = 2 - 2\rho^2$.

(b) For the vector field $\mathbf{F} = \langle 0, -z, y\rangle$, directly evaluate $\iint_S (\operatorname{curl} F)\cdot \mathbf{n}\, dA$ where $\mathbf{n}$ is unit normal vector field that points in the positive x-direction. **(5 points)**

$$\operatorname{curl}\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -z & y \end{vmatrix} = \langle 2,0,0\rangle$$

points in direction of $-\vec{n}$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 0 & -v\sin u & v\cos u \\ -4v & \cos u & \sin u \end{vmatrix} = \langle -v, -4v^2\cos u, -4v^2\sin u\rangle$$

$$\iint_S (\operatorname{curl}\vec{F})\,\vec{n}\,dA = \int_0^{2\pi}\int_0^1 \langle 2,0,0\rangle \cdot \langle +v, 4v^2\cos u, 4v^2\sin u\rangle\,dv\,du$$

$$= \int_0^{2\pi}\int_0^1 2v\,dv\,du = \int_0^{2\pi} v^2\Big|_{v=0}^{1} du$$

$$= \int_0^{2\pi} 1\,du = 2\pi$$

$$\iint_S (\operatorname{curl} F)\cdot \mathbf{n}\,dA = \boxed{2\pi}$$

(c) Check your answer in (b) using Stokes' Theorem. **(3 points)**

$$\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 0, -\sin t, \cos t\rangle \cdot \vec{r}\,'(t)\,dt$$

$\vec{r}(t) = (0, \cos t, \sin t)$

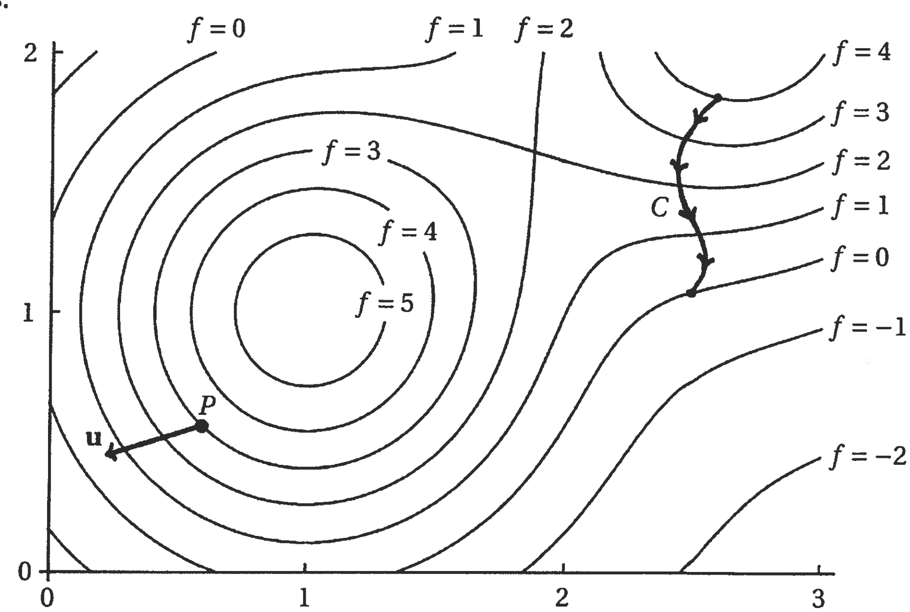
$\vec{r}\,'(t) = (0, -\sin t, \cos t)$

$$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt$$

$$= \int_0^{2\pi} 1\,dt = \boxed{2\pi.}$$

✓

13. Consider the function $f(x, y)$ on the rectangle $D = \{0 \le x \le 3 \text{ and } 0 \le y \le 2\}$ whose contours are shown below right. For each part, circle the best answer. **(1 point each)**



(a) The maximum value of f on D is:

0 3 6 9 DNE

(b) At P, the derivative $\dfrac{\partial^2 f}{\partial^2 y}$ is:

negative zero positive

(c) The value of $D_{\mathbf{u}} f(P)$ is:

negative zero positive

(d) The number of critical points of f in D which are saddles is:

0 1 2 3

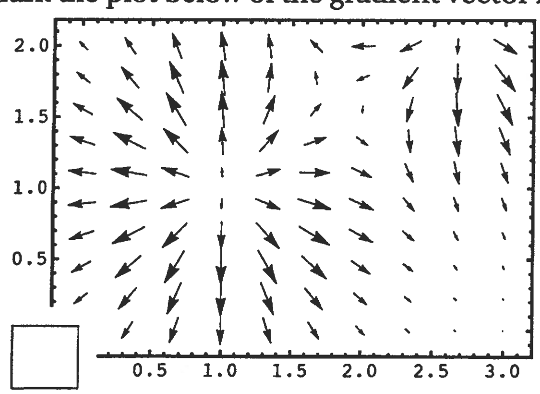
explanations on the next page

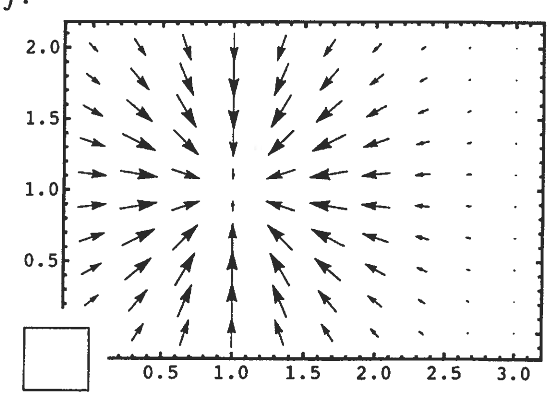
(e) The integral $\int_C f \, ds$ is: negative zero positive

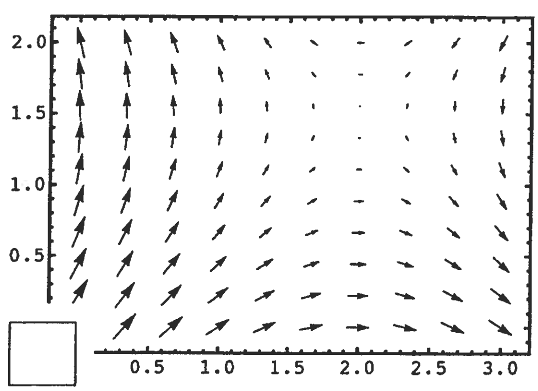
(f) The integral $\int_C \nabla f \cdot d\mathbf{r}$ is: −4 −2 0 2 4

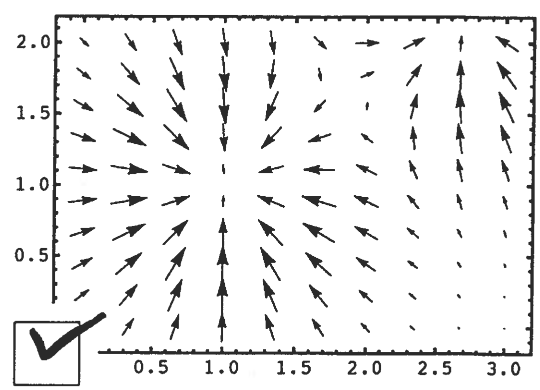
(g) The integral $\iint_D f \, dA$ is: −27 −18 −9 0 9 18 27

(h) Mark the plot below of the gradient vector field ∇f.









Problem 13

(a) The domain D is closed (it has $\leq$ in its definition) and it is bounded (the whole thing fits on the paper), and the function is continuous. The Extreme Value Theorem says there must be a maximum on D, so the answer is *not* DNE (the answer should be a number).

There is a contour line labeled $f = 5$, so the answer must be larger than 5. But if the answer was 9, we would see curves labeled $f = 6$ $f = 7$, and $f = 8$, but we don't. So the answer must be 6. (The maximum should occur inside the curve labeled $f = 5$, at the top of the hill.)

(b) In the y-direction, the curves get slightly further apart as you move further up the hill, and the curves get more vertical (parallel to y-axis), so the function is changing more slowly in that direction. Therefore the function is concave-down in the y-direction, so $\frac{\partial^2 f}{\partial y^2}$ is negative.

(c) In the direction of $\mathbf{u}$, the function is decreasing: $f(P) = 3$, and $\mathbf{u}$ is pointing towards the $f = 2$ contour. Since the function is decreasing in that direction, the directional derivative is negative.

(d) Saddle critical points occur at points where a contour curve intersects itself. This happens exactly once, on the $f = 2$ contour.

(e) f is positive everywhere along C (it's between 0 and 4), and the integral of a positive function is positive. Thus $\int_C f \, \mathrm{d}s$ is positive.

(f) By the Fundamental Theorem of Line Integrals:

$$\begin{aligned}\int_C \nabla f \cdot \mathrm{d}\mathbf{r} &= f\,(\text{end point of } C) - f\,(\text{start point of } C) \\ &= 0 - 4 \\ &= -4\end{aligned}$$

(g) If we divide D up into 6 pieces (into thirds with vertical cuts and into half with a horizontal cut), then $\iint_D f \, \mathrm{d}A$ is equal to the sum of the integrals of each piece. Then we estimate the integral of each piece, using

$$\iint_{\text{piece}} f \, \mathrm{d}A = (\text{average value of } f \text{ on piece})\,(\text{area of piece})\,.$$

We estimate the average on each piece by taking the value of the function at the midpoint of each piece (the points $(0.5, 0.5)$, $(1.5, 0.5)$, and so on). The estimates come from eyeballing the graph. Everyone's eyes are different, so your estimates may be different by a little but not much.

$$\begin{aligned} f\,(0.5, 1.5) &\approx 2; \\ f\,(1.5, 1.5) &\approx 2.5; \\ f\,(2.5, 1.5) &\approx 2; \\ f\,(0.5, 0.5) &\approx 2.5; \\ f\,(1.5, 0.5) &\approx 2.5; \\ f\,(2.5, 0.5) &\approx -1.5; \\ \text{Area of each piece} &= 1; \end{aligned}$$

Then:

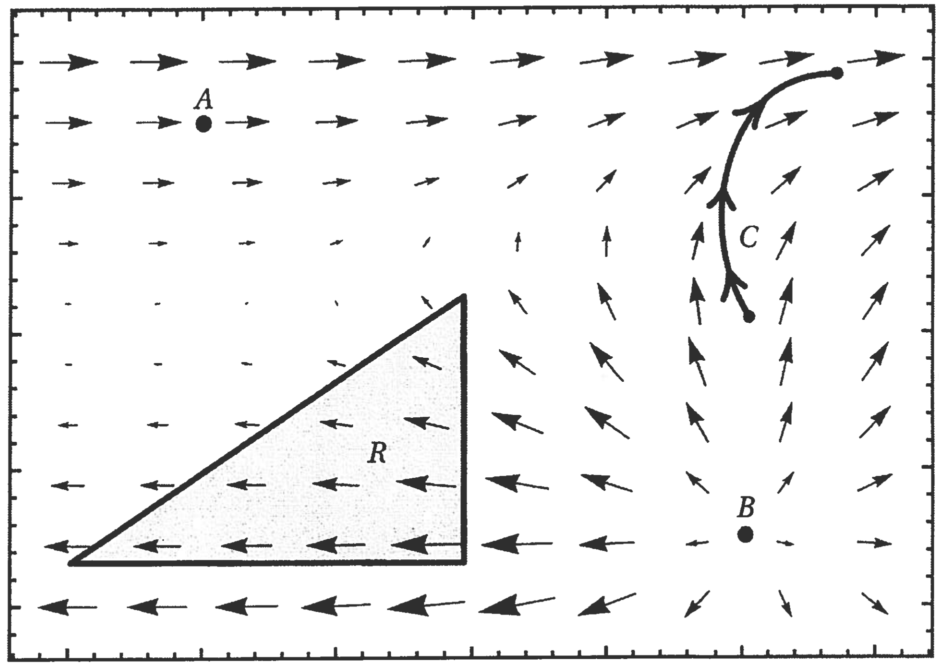
$$\iint_D f \, \mathrm{d}A \approx (2)(1) + (2.5)(1) + (2)(1) + (2.5)(1) + (2.5)(1) + (-1.5)(1) = 10.$$

9 is the answer closest to our estimate.

(h) The contour curves on the left of the contour plot, that look like concentric circles, indicate there is a hill there. The gradient always points in the direction of increase, so the gradient should point towards the top of the hill. The plots on the right do this.

There is also a hill near the top-right corner of the domain, so the gradient should point towards the top of that hill too. The top-right plot does not do this, and the bottom-right plot does, so the bottom-right is correct.

14. Consider the vector field $\mathbf{F}$ on $\mathbb{R}^2$ shown below right. For each part, circle the best answer. **(1 point each)**



(a) The line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is:

negative	zero	positive

(b) At A, the vector $\operatorname{curl}\mathbf{F}$ is:

$\langle 1,0,0\rangle$	$\langle 0,0,-1\rangle$	$\langle 0,0,1\rangle$

(c) At B, the divergence $\operatorname{div}\mathbf{F}$ is:

negative	zero	positive

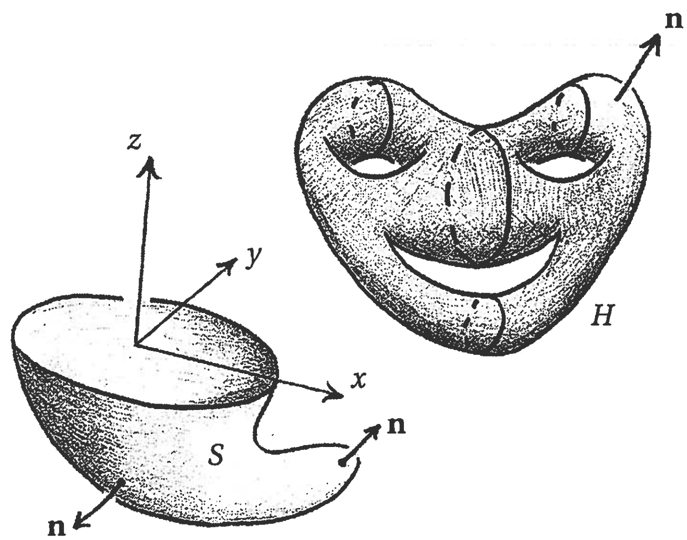
(d) If $\mathbf{F} = \langle P, Q\rangle$, then $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ is:

negative	zero	positive

(e) The vector field $\mathbf{F}$ is conservative:

True	False

15. Consider the surfaces S and H show below right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. For each part, circle the best answer.



(a) For $\mathbf{F} = \langle yz,\ xz + x,\ z\rangle$, the integral $\iint_H \mathbf{F}\cdot\mathbf{n}\, dA$ is:

negative	zero	positive

(1 point)

(b) The flux of $\operatorname{curl}\mathbf{F} = \langle -x, y, 1\rangle$ through H is:

negative	zero	positive

(1 point)

(c) The integral $\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}$ is:

-2π	$-\pi$	0	π	2π

(2 points)

(d) For $\mathbf{G} = \langle y,\ z,\ 2\rangle$, the integral $\iint_S \mathbf{G}\cdot\mathbf{n}\, dA$ is:

-2π	$-\pi$	0	π	2π

(2 points)

explanations on following pages

SCRATCH WORK MAY GO HERE.

Problem 14

(a) The vectors in $\mathbf{F}$ point in the same general direction as the curve C (the angle between $\mathbf{F}$ and the direction of C is acute). Thus the dot product $\mathbf{F} \cdot \mathbf{dr}$ is positive, so the integral is positive.

(b) The curl of a vector field living in $\mathbb{R}^2$ (two-dimensions) is always in the z direction, so it cannot be $\langle 1, 0, 0 \rangle$.

If $\mathbf{F} = \langle P, Q \rangle$, note that Q is always 0 in that part of the graph, so $\frac{\partial Q}{\partial x} = 0$. Since P is increasing as y increases, $\frac{\partial P}{\partial y}$ is positive. Thus:

$$\begin{aligned} z\text{-coord of curl}\,(\mathbf{F}) &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ &= 0 - (\text{positive}) \\ &= \text{negative} \end{aligned}$$

The only answer with a negative z-coordinate is $\langle 0, 0, -1 \rangle$.

Another way to answer this is by the right-hand rule. If you curl your fingers in the direction $\mathbf{F}$ is rotating, your thumb points in the direction of curl $(\mathbf{F})$. $\mathbf{F}$ is turning clockwise, so your thumb will point into the page (which is in the negative z direction. (It may not look like it's rotating since the vectors all point horizontally, but consider this: let $\mathbf{F}$ be the flow of water, and drop a stick in the water at A. Then the stick will turn clockwise since the top is being pushed harder than the bottom.)

(c) P goes from negative to positive as you move in the x direction, so P is increasing. Thus $\frac{\partial P}{\partial x}$ is positive. Also, Q goes from negative to positive as you move in the y direction, so $\frac{\partial Q}{\partial y}$ is positive. Then $\operatorname{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ is positive.

Intuitively, $\mathbf{F}$ flows away from B, so B is acting as a source, which means the divergence is positive.

(d) The field is turning clockwise in R (by right-hand rule; see explanation for part (b) to see why), so the curl is pointing in the negative z-direction. Since $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is the z-coordinate of curl, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is negative and so the integral is negative.

(e) Conservative vector fields always have zero curl, and $\mathbf{F}$ has non-zero curl. (For example, no matter what your answer for (b) is, it's not zero.)

Another reason is that since $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ is non-zero, and $\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{\partial R} \mathbf{F} \cdot \mathbf{dr}$ by Green's Theorem, then $\mathbf{F}$ is not path-independent since $\oint_{\partial R} \mathbf{F} \cdot \mathbf{dr}$ is non-zero.

Problem 15

(a) We have $\operatorname{div}(\mathbf{F}) = 0+0+1 = 1$. Let R be the three-dimensional inside of the surface H. By the Divergence Theorem, $\iint_H \mathbf{F} \cdot \mathbf{n}\, dA = \iiint_R \operatorname{div}(\mathbf{F})\, dV = \iiint_R 1\ dV = \text{Volume}(R) > 0$.

(b) By Stokes' Theorem, $\iint_H \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial H} \mathbf{F} \cdot d\mathbf{r}$. But H has no boundary, so the integral of anything over the boundary of H is zero.

(c) By Stokes' Theorem, $\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}$. The boundary of S is the unit circle in the xy-plane; the normal vector of S is downward, so the circle is oriented clockwise (as viewed from above), by the right-hand rule.

There are two options to go from here. The first: by Stokes' Theorem, any other surface with the same boundary and orientation has the same flux integral of $\operatorname{curl}(\mathbf{F})$. The unit disk in the xy-plane, pointing downwards, is one such surface. Then:

$$\begin{aligned}
\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \iint_{\text{disk}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \\
&= \iint_{\text{disk}} \langle -x, y, 1\rangle \cdot \langle 0, 0, -1\rangle\, dA \\
&= \iint_{\text{disk}} -1\ dA \\
&= -\text{Area}(\text{disk}) \\
&= -\pi
\end{aligned}$$

The other option is to calculate the line integral over the boundary directly. $\mathbf{r}(t) = \langle \sin t, \cos t, 0\rangle$, $0 \le t \le 2\pi$ is a parametrization of the circle, going clockwise. Also, $\mathbf{r}'(t) = \langle \cos t, -\sin t, 0\rangle$. Then:

$$\begin{aligned}
\iint_S \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^{2\pi} \langle (\cos t)(0), (\sin t)(0) + \sin t, 0\rangle \cdot \langle \cos t, -\sin t, 0\rangle\, dt \\
&= \int_0^{2\pi} -\sin^2(t)\, dt \\
&= -\pi
\end{aligned}$$

(d) Make S into a solid three-dimensional shape R by adding the unit disk, oriented upwards, so that the orientation of the boundary ∂R is outwards. Then $\iint_{\partial R} =$

$\iint_S + \iint_{\text{disk}}$. Then apply Divergence Theorem:

$$\begin{aligned}
\iint_S \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}A &= \iint_{\partial R} \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}A - \iint_{\text{disk}} \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}A \\
&= \iiint_R \operatorname{div}(\mathbf{G})\,\mathrm{d}V - \iint_{\text{disk}} \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}A \\
&= \iiint_R 0\ \mathrm{d}V - \iint_{\text{disk}} \langle y, z, 2\rangle \cdot \langle 0,0,1\rangle\,\mathrm{d}A \\
&= 0 - \iint_{\text{disk}} 2\ \mathrm{d}A \\
&= -2\,\text{Area}\,(\text{disk}) \\
&= -2\pi
\end{aligned}$$