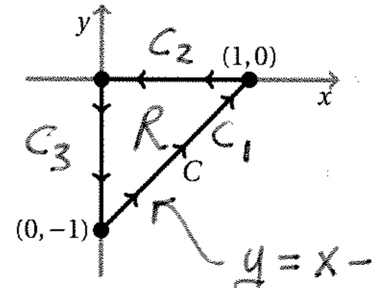
1. Let C denote the curve pictured at right, with the orientation shown.



(a) For $\mathbf{F}(x, y) = \langle xy, 0\rangle$, compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ directly. **(3 points)**

Note that $\vec{F} = \langle 0,0\rangle$ on C_2 and C_3,

so $$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} xy\,dx$$

$$= \int_0^1 t(t-1)\,dt = \int_0^1 t^2 - t\,dt = \frac{t^3}{3} - \frac{t^2}{2}\Big|_{t=0}^{1} = -\frac{1}{6}$$

$\left.\begin{array}{l} x = t \\ y = t-1 \end{array}\right\}$ param of C_1

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = -\frac{1}{6}}$$

(b) Check your answer to part (a) using Green's Theorem. **(3 points)**

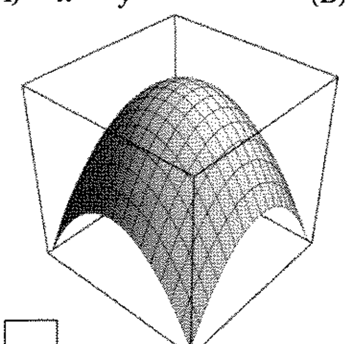
$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = \int_0^1\int_{x-1}^0 -x\,dy\,dx$$

$$= \int_0^1 -x + x^2\,dx = -\frac{x^2}{2} + \frac{x^3}{3}\Big|_{x=0}^{1} = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$$

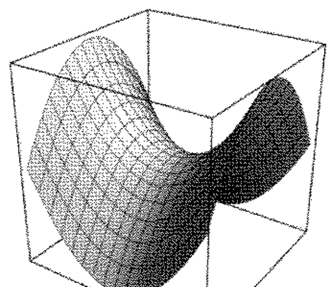
2. For each function label its graph from among the options below: **(2 points each)**

see next page for explanations

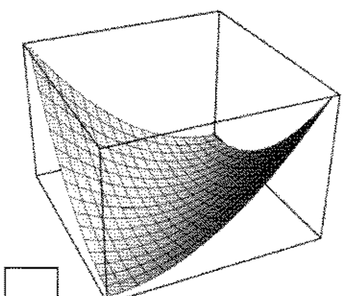
(A) $x^2 - y^2$ (B) $\cos(xy)$ (C) $e^{-(x^2+y^2)}$



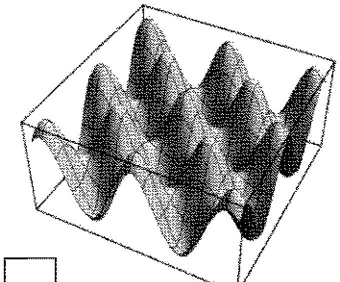
☐



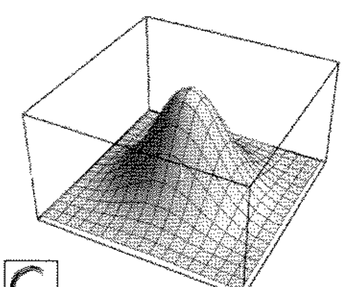
A



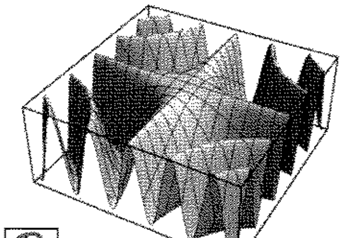
☐



☐



C



B

(2) **Answer:**

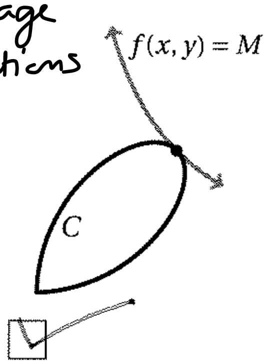
(A) We see that if we set $x = 0$, then the function becomes $-y^2$, which is a parabola openning downward. Similarly, if we set $y = 0$, then the function becomes x^2, which is a parabola opening upwards. The only picture that matches this one is the upper middle one.

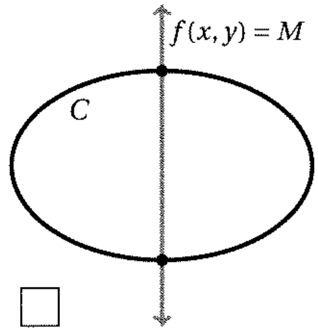
(B) We see that along $x = 0$ and $y = 0$, the function has a constant value of 1. On the other hand, along $y = x$ and $y = -x$, the function has a value $\cos(2x)$. The only picture that matches this one is the lower right one.

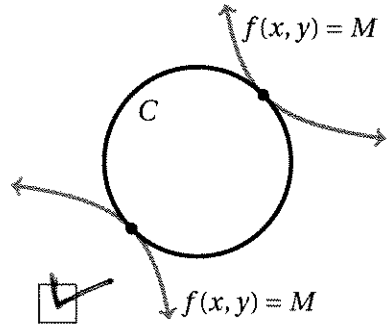
(C) If we rewrite this function in terms of polar coordinates, it becomes e^{-r^2}. Since this does not depend on θ, the graph should be rotationally symmetric. If $r = 0$, the function has a value 1. If r is a very large positive number, the function will approach zero. The only picture that matches this one is the lower middle one.

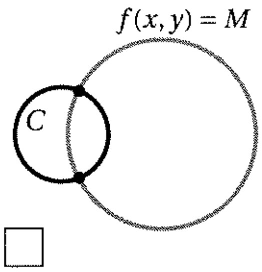
3. (a) Each picture below depicts both (i) a constraint curve C defined by $g(x, y) = 1$ for a function $g(x, y)$, and (ii) a level curve $f(x, y) = M$ of a function $f(x, y)$. Mark the boxes of **all and only those pictures** for which M could be the maximum value of $f(x, y)$ subject to the constraint $g(x, y) = 1$. [In every picture, you should assume that ∇f is always nonzero.] **(2 points)**

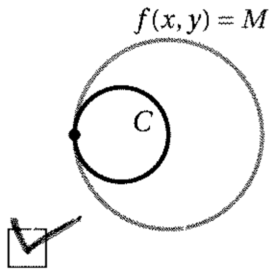
see next page for explanations

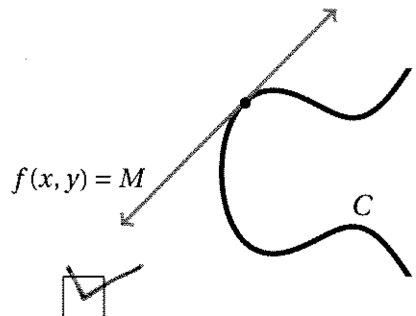












(b) Suppose a function $f(x, y)$ attains its minimum value, subject to the constraint $\overbrace{2x^2 + 2xy^2 + y^3}^{g} = 5$, at $(x, y) = (1, 1)$. Assuming that $\nabla f(1, 1) \neq \langle 0, 0\rangle$, find a nonzero vector $\mathbf{v}$ parallel to $\nabla f(1, 1)$. **(3 points)**

At a minimum, we will have the lagrange condition:

$$\nabla f(1,1) = \lambda \nabla g(1,1) = \lambda \langle 6, 7\rangle$$

$$\nabla g = \langle 4x + 2y^2, 4xy + 3y^2\rangle$$

$\mathbf{v} = \langle\ 6\ ,\ 7\ \rangle$

4. Suppose $f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ has the table of values and partial derivatives shown at right. For $x(s, t) = s + 2t$ and $y(s, t) = s^2 - t$, let $F(s, t) = f\big(x(s, t), y(s, t)\big)$ be their composition with f. Compute $\dfrac{\partial F}{\partial t}(2, 1)$. **(3 points)**

(x, y)	$f(x, y)$	$\dfrac{\partial f}{\partial x}$	$\dfrac{\partial f}{\partial y}$
(2, 1)	0	7	6
(2, −1)	-12	7	-1
(4, 3)	7	3	1
(5, 3)	19	-8	5

$$\frac{\partial F}{\partial t}(2,1) = \frac{\partial f}{\partial x}\big(\overset{4}{x(2,1)}, \overset{3}{y(2,1)}\big) \frac{\partial x}{\partial t}(2,1) + \frac{\partial f}{\partial y}\big(x(2,1), y(2,1)\big) \frac{\partial y}{\partial t}(2,1)$$

$$= 3 \cdot 2 + 1 \cdot (-1) = 5$$

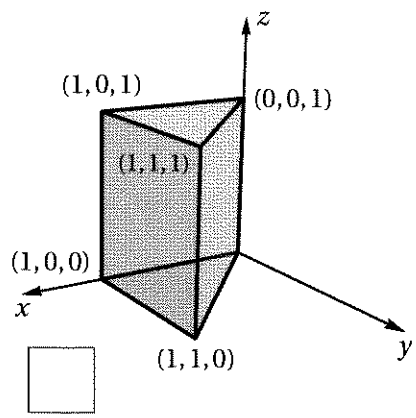
$\dfrac{\partial F}{\partial t}(2, 1) = 5$

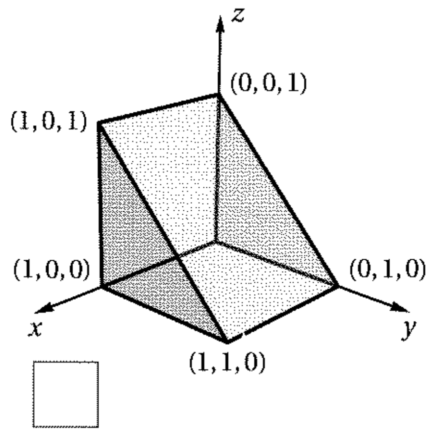
(3) **Answer:** The level curve corresponding to the maximum value in a constrained optimization problem must be tangent to the constraint curve.

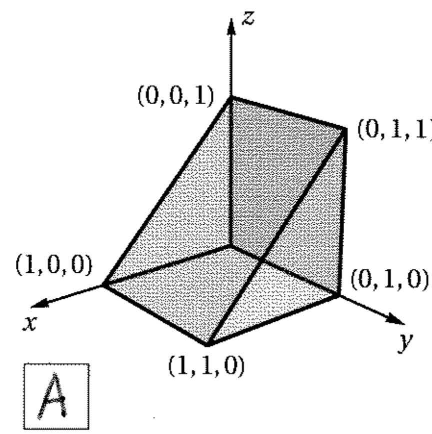
5. For each of the integrals   see next page for explanation

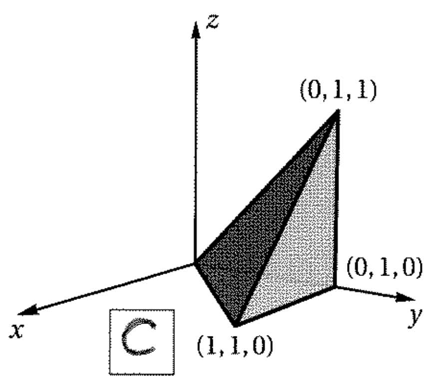
(A) $\int_0^1 \int_0^1 \int_0^{1-x} f(x,y,z)\, dz\, dy\, dx$ (B) $\int_0^1 \int_0^1 \int_0^{y} f(x,y,z)\, dx\, dy\, dz$ (C) $\int_0^1 \int_x^1 \int_0^{y-x} f(x,y,z)\, dz\, dy\, dx$

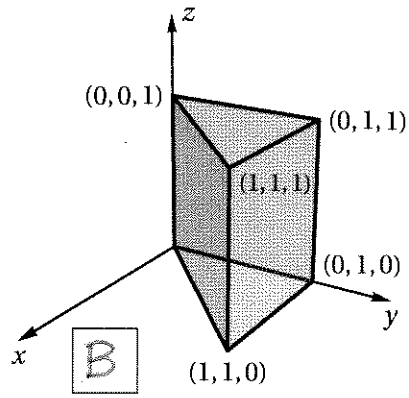
label the solid corresponding to the region of integration below. **(1 point each)**

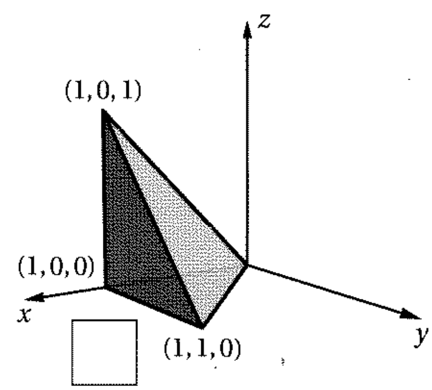










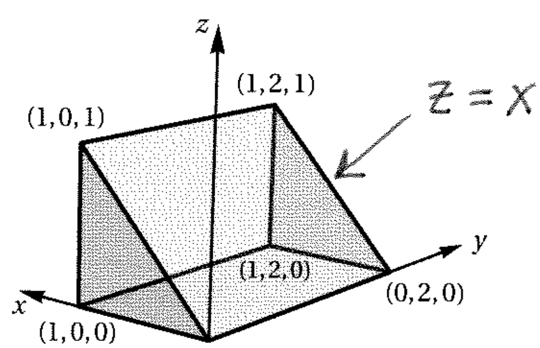


6. Compute the mass of solid region E shown at right if the mass density is $\rho(x,y,z) = z$. **(4 points)**

$$\int_0^1 \int_0^2 \int_0^x z\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^2 \frac{z^2}{2}\Big|_0^x\, dy\, dx$$

$$= \int_0^1 \int_0^2 \frac{x^2}{2}\, dy\, dx = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_{x=0}^{x=1}$$



Mass = 1/3

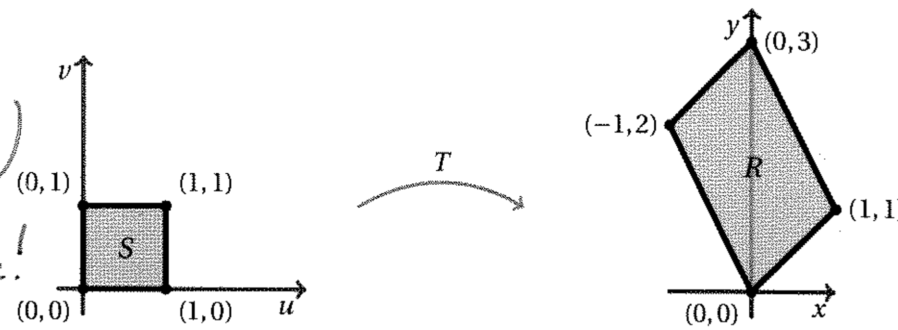
(5) **Answer:**

(A) If we project this shape onto the xy-plane, we will have the unit square, since the x-bounds and the y-bounds are both 0 to 1. Additionally, one of the z-bounds is $z = 1 - x$. The only option that has these characteristics is the top right one.

(B) If we project this shape onto the yx-plane, we will have the unit square, since the y-bounds and the z-bounds are both 0 to 1. Additionally, the x-bounds are $x = 0$ and $x = y$. The only option that has these characteristics is the bottom middle one.

(C) If we project this shape onto the xy-plane, we will have the triangle bounded by $x = 0$, $x = y$, and $y = 1$. Additionally, the top z-bound is $z = y - x$. Thie only option that has these characteristics is the bottom left one.

7. (a) Let R be the region shown below right. Find a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [0,1] \times [0,1]$ to R. **(3 points)**



Use linear trans

$T(u,v) = (au + bv,\ cu + dv)$

where

our choice!

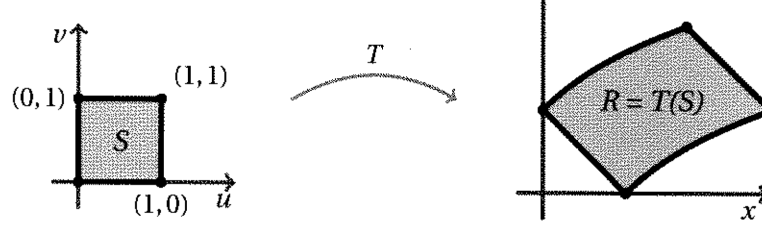
$T(1,0) = (a,c) = (1,1)$

and

$T(0,1) = (b,d) = (-1,2)$

$T(u,v) = \langle\ u - v\ ,\ u + 2v\ \rangle$

(b) Consider the transformation $T(u,v) = \left(e^u - v, u + v\right)$ whose behavior is depicted below. Compute $\iint_R 3\ dA$ via an integral over S. **(3 points)**



$J = \begin{pmatrix} e^u & -1 \\ 1 & 1 \end{pmatrix}$

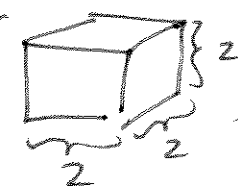
$|\det J| = e^u + 1$

$\iint_R 3\ dA = \underbrace{\int_0^1\int_0^1}_{\text{int over } S} 3 \cdot |\det J|\ du\, dv = 3\int_0^1\int_0^1 e^u + 1\ du\,dv$

$= 3 \cdot \int_0^1 e^u + u \Big|_{u=0}^{u=1} dv = 3\int_0^1 e\ dv$

$\iint_R 3\ dA = 3e$

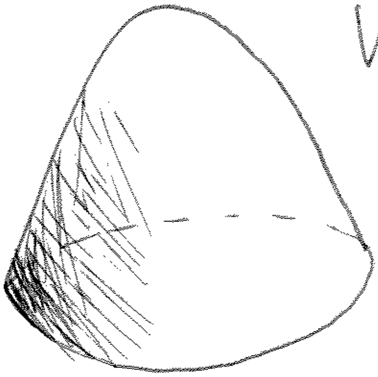
8. Let S be the surface in $\mathbb{R}^3$ which is the boundary of the solid cube $D = \{-1 \le x \le 1, -1 \le y \le 1, -1 \le z \le 1\}$. For $\mathbf{F}(x,y,z) = \langle yz^2 + e^z + x,\ ze^z + x + y,\ xe^x + xy + z\rangle$, compute $\iint_S \mathbf{F} \cdot \mathbf{n}\ dS$ by any valid method, where $\mathbf{n}$ is the outward-pointing unit normal vector field. **(4 points)**

$= \iiint_{\text{Cube}} \operatorname{div} \vec{F}\, dV = \iiint_{\text{Cube}} 3\ dV = 3\,\text{Vol}($  $)$

$= 3 \cdot 2^3 = 24$

$\iint_S \mathbf{F} \cdot \mathbf{n}\ dS = 24$

9. Consider the region R below the surface $z = 1 - x^2 - y^2$ and above the xy-plane. Compute the volume of R. **(5 points)**



$$\text{Volume} = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^1 r(1-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta$$

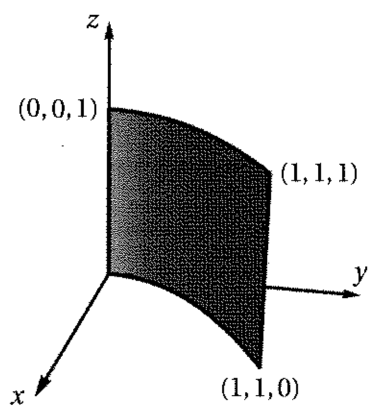
Use cylindrical coordinates

$$= \int_0^{2\pi} \left.\frac{r^2}{2} - \frac{r^4}{4}\right|_{r=0}^{r=1} d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \pi/2$$

Volume = $\pi/2$

10. For each surface S in parts (a) and (b) give a parameterization $\mathbf{r}\colon D \to S$. Be sure to explicitly specify the domain D and call your parameters u and v.

(a) The portion of the surface $x = y^2$ shown at left. **(2 points)**

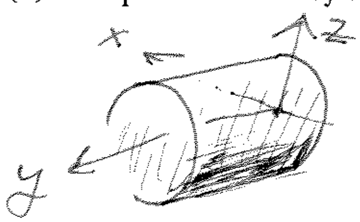


Take $u = y$ and $v = z$

$D = \{\ 0 \le u \le 1$ and $0 \le v \le 1\ \}$

$\mathbf{r}(u, v) = \langle\ u^2\ ,\ u\ ,\ v\ \rangle$

(b) The portion of the cylinder $x^2 + z^2 = 1$ between the planes $y = 0$ and $y = 2$. **(3 points)**



Params: $u = y$ (along cylinder)
$v = \theta$, angle around y axis

$D = \{\ 0 \le u \le 2$ and $0 \le v \le 2\pi\ \}$

$\mathbf{r}(u, v) = \langle\ \cos v\ ,\ u\ ,\ \sin v\ \rangle$

(c) Let M be the surface in part (b). Is the surface integral $\iint_M y\,dS$: negative zero positive

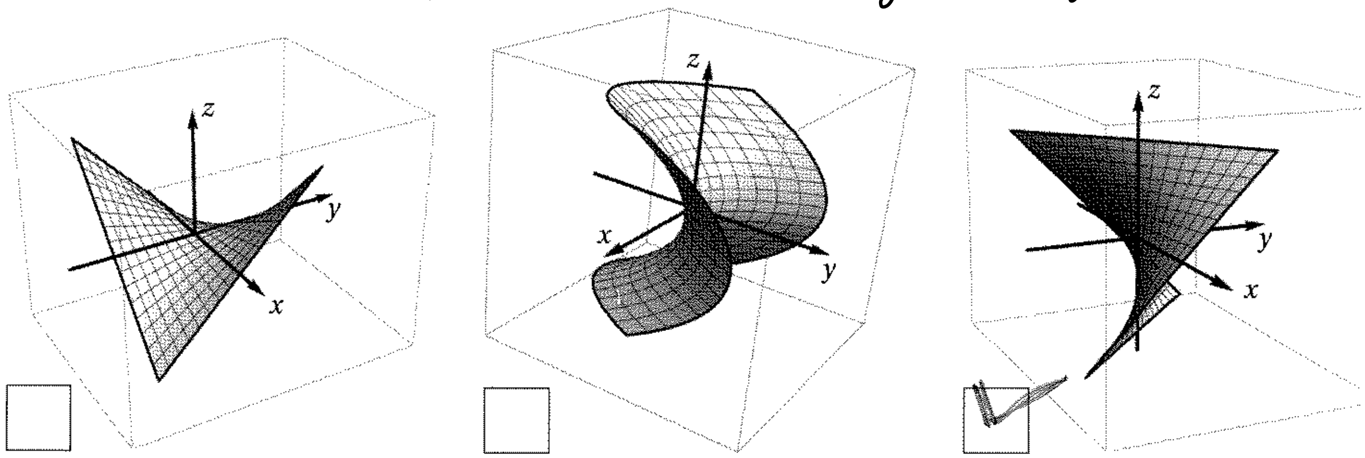
Circle your answer. **(1 point)**

↳ see next page for explanation

(10.c) **Answer:** The surface M lives in the space $y \geq 0$. That, is, y is positive on most of M, and zero at the rest of the points. This tells us the integral is positive.

11. Let S be the surface parameterized by $\mathbf{r}(u,v) = \langle u,\ uv,\ v\rangle$ for $-1 \le u \le 1$ and $-1 \le v \le 1$.

(a) Mark the picture of S below. **(2 points)** see next page for explanation



(b) Completely setup, but do not evaluate, the surface integral $\iint_S x^2\, dS$. **(5 points)**

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & v & 0 \\ 0 & u & 1 \end{vmatrix} = \langle v, -1, u\rangle$$

$$\iint_S x^2\, dS = \int_{-1}^{1}\int_{-1}^{1} u^2\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} u^2 \sqrt{1+u^2+v^2}\, du\, dv$$

cor to $u=0$ and $v=0$

(c) Find the tangent plane to S at $(0,0,0)$. [You *must* show work that justifies your answer.] **(2 points)**

Normal is $\vec{r}_u \times \vec{r}_v$ at $(0,0) = \langle 0, -1, 0\rangle$

$\Rightarrow$ plane is $y = 0$

Equation: $\boxed{0}\,x + \boxed{1}\,y + \boxed{0}\,z = \boxed{0}$

(11.a) **Answer:** If we look at the edges of the domain for the parametrization, we see that we'll get straight lines. For example, on the edge $u = 1$, the parametrization is

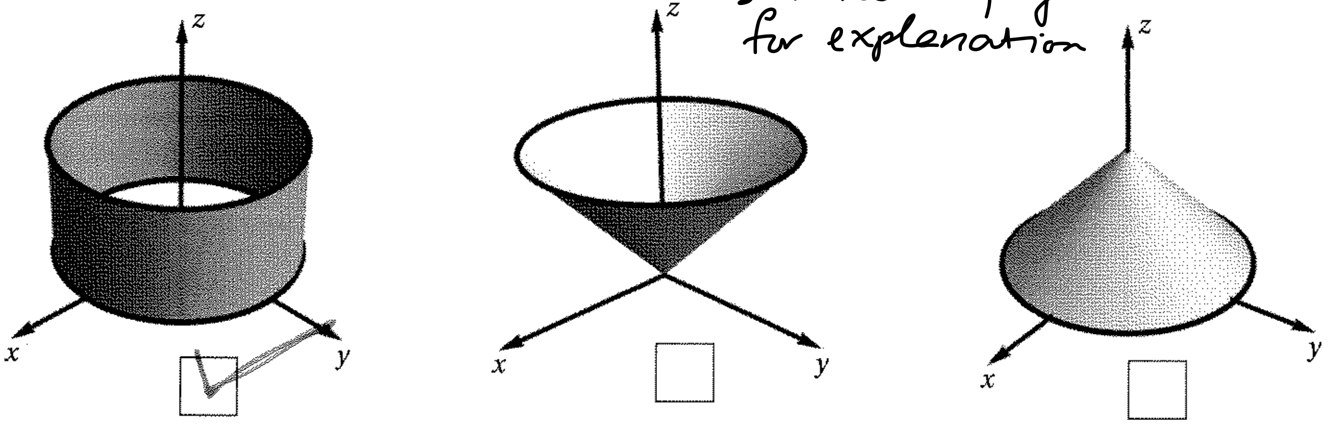
$$\mathbf{r}(1, v) = \langle 1, v, v \rangle$$

which is a straight line. This rules out the middle option. Additionally, if we set $u = 0$, the parametrization is $\mathbf{r}(0, v) = \langle 0, 0, v \rangle$, which goes along the z-axis. This rules out the left option.

12. Consider the surface S parameterized by $\mathbf{r}(u,v) = \langle \cos u,\ \sin u, v\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$.

(a) Mark the picture of S below. **(2 points)**

see next page for explanation



(b) Consider the vector field $\mathbf{F} = \langle yz, -xz, 1\rangle$ which has $\operatorname{curl}\mathbf{F} = \langle x, y, -2z\rangle$. Directly evaluate $\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\, dS$ via the given parameterization, where $\mathbf{n}$ is the outward normal vector field. **(4 points)**

$$= \int_0^1 \int_0^{2\pi} \langle \cos u, \sin u, -2v\rangle \cdot \overbrace{\langle \cos u, \sin u, 0\rangle}^{r_u \times r_v}\, dv\, du$$

$$= \int_0^1 \int_0^{2\pi} \overbrace{\cos^2 u + \sin^2 u + 0}^{1}\, dv\, du$$

$$= 2\pi$$

$$\vec r_u \times \vec r_v = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \langle \cos u, \sin u, 0\rangle$$

$\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\, dS = 2\pi$

(c) Check your answer in (b) using Stokes' Theorem. **(4 points)**

Param for C_1: $\vec r(t) = (\cos t, \sin t, 0)$

$$\int_{C_1} \vec F \cdot d\vec r = \int_0^{2\pi} \langle 0,0,1\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$

$$\int_{C_2} \vec F \cdot d\vec r = \int_0^{2\pi} \overbrace{\langle \cos t, -\sin t, 1\rangle \cdot \langle \cos t, -\sin t, 0\rangle}^{\cos^2 t + \sin^2 t + 0 = 1}\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi.$$

Matches!

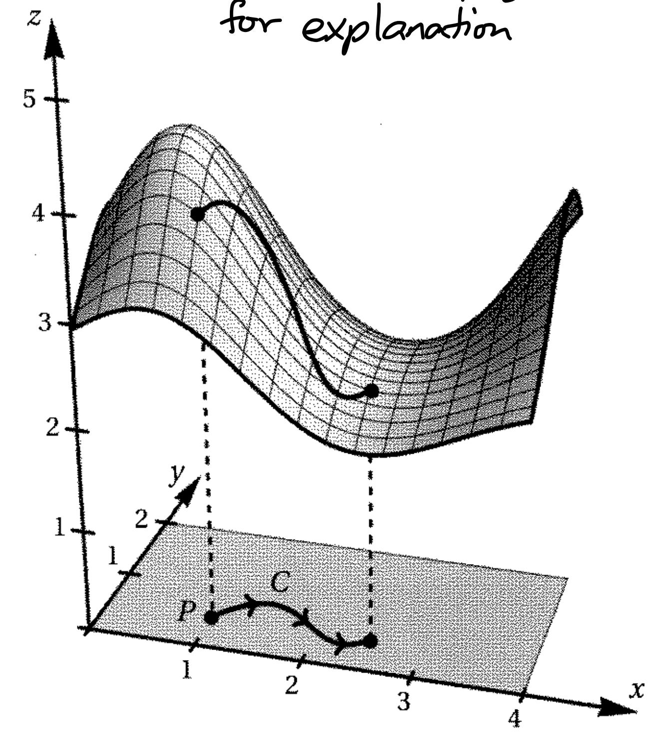
$\vec r_2(t) = (\sin t, \cos t, 1)$

$\boxed{\text{So flux of } \operatorname{curl}\vec F = \int_{\partial S} \vec F \cdot d\vec r = 0 + 2\pi = \boxed{2\pi}}$

(12.a) **Answer:** Note that the components of this parametrization satisfy $x^2 + y^2 = 1$. Thus, the surface must be a part of the cylinder around the z-axis of radius 1. The only option is the left one.

13. Consider the function $f(x,y)$ on the rectangle $D = \{0 \le x \le 4 \text{ and } 0 \le y \le 2\}$ whose graph is shown below right. For each part, circle the best answer. **(1 point each)**

see next page for explanation



(a) At the point $P = (1, 0.5)$ is $\dfrac{\partial f}{\partial y}$:

negative zero positive

(b) At P is $\dfrac{\partial^2 f}{\partial x^2}$:

negative zero positive

(c) How many critical points does f have in the *interior* of D?

0 1 2 3 4

(d) The integral $\iint_D f(x,y)\, dA$ is:

negative zero positive

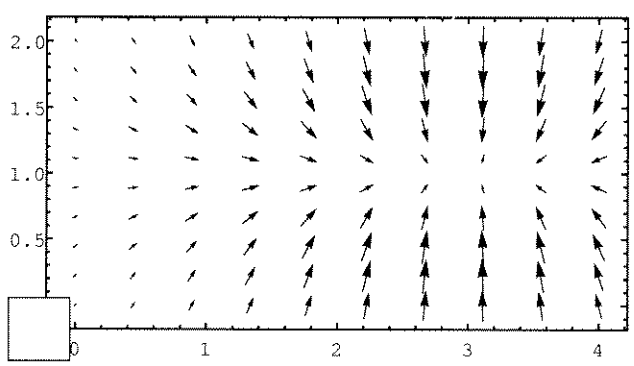
(e) For the curve C shown, the line integral $\int_C \nabla f \cdot d\mathbf{r}$ is:

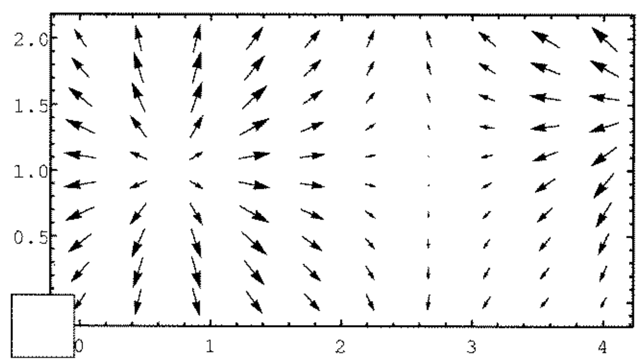
-3 -1.5 0 1.5 3

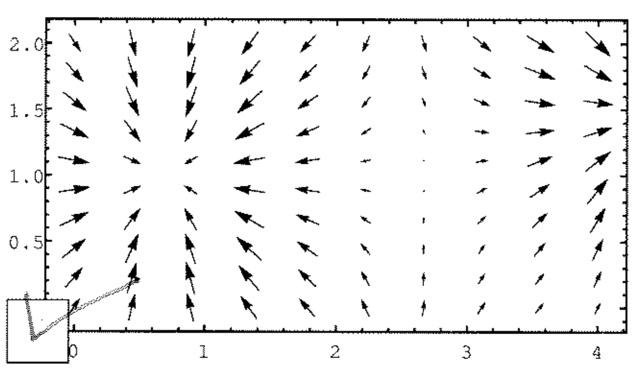
(f) The line integral $\int_C f\, ds$ is:

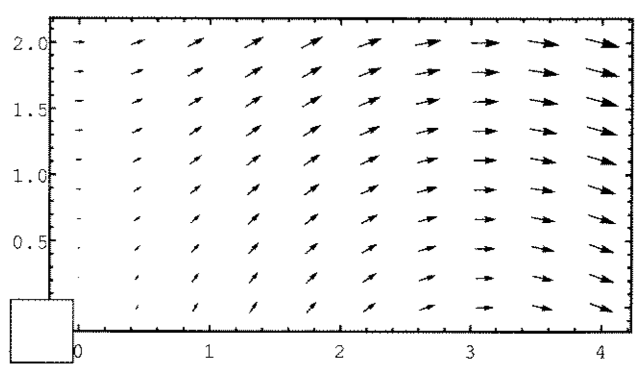
negative zero positive

(g) Mark the plot of the vector field ∇f.









(13) **Answer:**

(a) If we move along the surface starting from the point P in the positive y direction, we will move uphill.

(b) In the y direction, the function is "concave down".

(c) The critical points on the interior of D are maxima, minima, and saddlepoints. There is exactly one maximum and one saddlepoint.

(d) Since the integrand is $f(x, y)$, which is positive everywhere on D, this integral will be positive.

(e) By the Fundamental Theorem of Line Integrals, we have that this integral is $f(Q) - f(P)$, where Q is the point where C ends. We estimate these values are $f(P) = 4$ and $f(Q) = 2.5$.

(f) Since the integrand is $f(x, y)$, which is positive everywhere on C, this integral will be positive.

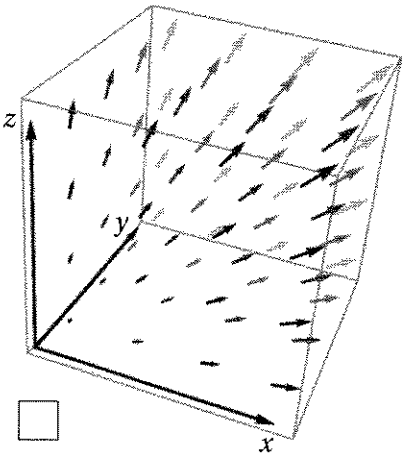
(g) Remember, the gradient vector field points in the direction of steepest ascent. In particular, around a maximum, it will point towards the maximum. In this case, we want a vector field that, around P, points toward P. This is only satisfied by the bottom left option.

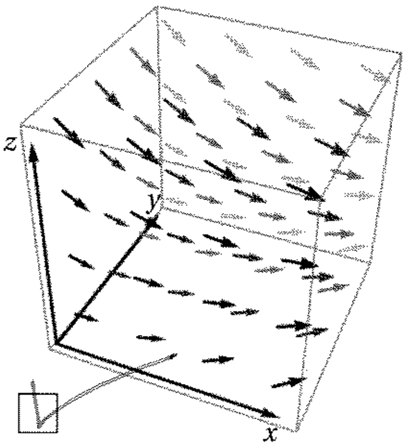
see next page for explanation

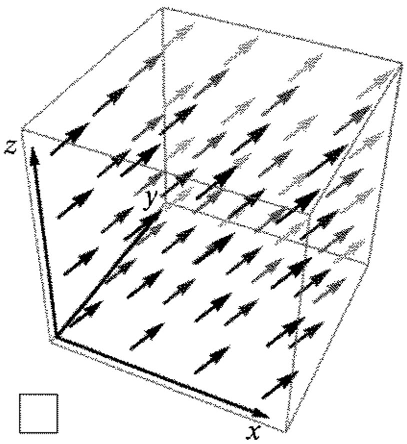
14. For each problem, circle the best answer. (**1 point each**)

(a) Consider the vector field $\mathbf{F} = \langle 1,\ x,\ -z \rangle$. The vector field $\mathbf{F}$ is: | conservative | not conservative |

(b) Mark the plot of $\mathbf{F}$ on the region where each of x, y, z is in $[0, 1]$:

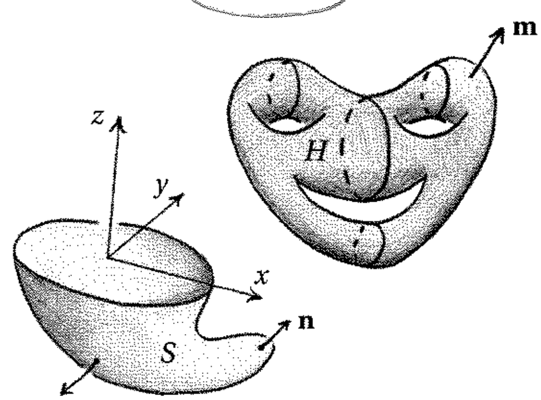






(c) For the leftmost vector field in part (b) is the divergence: | negative | zero | positive |

Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. Let $\mathbf{G} = \langle x,\ y,\ z \rangle$.



(d) The flux $\iint_H \mathbf{G} \cdot \mathbf{m}\, dS$ is:

| negative | zero | positive |

(e) The flux $\iint_S \mathbf{G} \cdot \mathbf{n}\, dS$ is: | negative | zero | positive |

(f) The flux $\iint_S (\operatorname{curl} \mathbf{G}) \cdot \mathbf{n}\, dS$ is: | negative | zero | positive |

(14) **Answer:**

(a) If the vector field $\mathbf{F}$ were conservative, we would have $\operatorname{curl} \mathbf{F} = 0$. Since $\operatorname{curl} \mathbf{F} \neq 0$, we have that $\mathbf{F}$ is not conservative.

(b) The option on the right is a constant vector field, which is inconsistent with $\mathbf{F}$. The option on the left has a changing x-component, which is again inconsistent with $\mathbf{F}$.

(c) If we imagine this vector field as describing the flow of a fluid, we see that it is flowing out of the origin, which means the divergence is positive.

(d) By Divergence Theorem, this integral calculates $3V$, where V is the volume of the interior of H.

(e) Let D be the unit disk in the xy-plane, and let E be the region bounded by D and S. Divergence Theorem tells us

$$\iint_S \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}S = 3 \iiint_E \mathrm{d}V - \iint_D \mathbf{G} \cdot \mathbf{n} \, \mathrm{d}S.$$

Now the last integral in this equation is zero since, on D, the normal vector is perpendicular to $\mathbf{G}$.

(f) Since $\operatorname{curl} \mathbf{G} = 0$, this integral is zero.