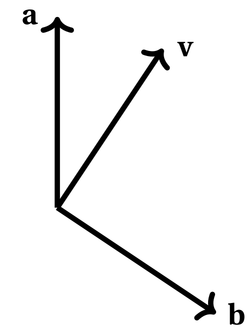
1. Let **v**, **a**, and **b** be the vectors (in the plane of the paper) drawn at right, all of which have length 1. Let **w** be a vector of length 2 pointing directly out of the paper. Which of the following vectors is $\mathbf{v} \times \mathbf{w}$? **(2 points)**

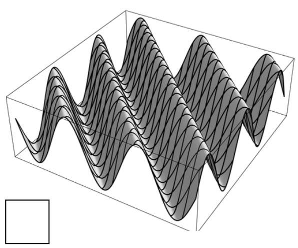


explanation on following page

$-2\mathbf{b}$	$-\mathbf{b}$	$\mathbf{b}$	$2\mathbf{b}$	$\langle 0,0,0\rangle$	$-2\mathbf{a}$	$-\mathbf{a}$	$\mathbf{a}$	$2\mathbf{a}$

2. A function $u\colon \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if it satisfies Laplace's equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$. Check the box next to the unique graph below that corresponds to a harmonic function. **(2 points)**

explanation on following page



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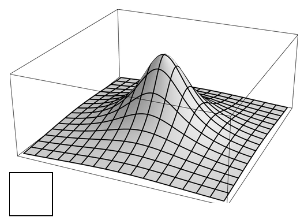

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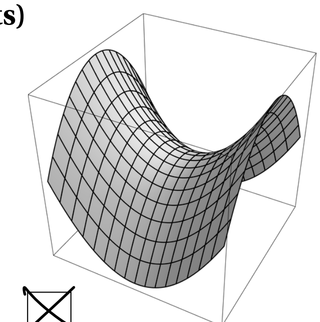


☒

3. Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$. Suppose that $f(x,y) \to 3$ as (x,y) approaches $(0,1)$ along every line of the form $y = kx + 1$. What can you say about the limit $\lim_{(x,y)\to(0,1)} f(x,y)$? Check the box next to the correct statement. **(2 points)**

- ☐ It exists and is equal to 3.
- ☒ We cannot determine if the limit exists, but if it does, the limit is 3.
- ☐ It does not exist.

explanation on following page

4. Suppose we know the following data about $g : \mathbb{R}^3 \to \mathbb{R}$.

(x,y,z)	$g(x,y,z)$	$g_x(x,y,z)$	$g_y(x,y,z)$	$g_z(x,y,z)$
$(3,3,1)$	30	6	4	5
$(0.5,0.1,0)$	1	2	6	4

Circle the best estimate for $g(3.5,3.1,1)$:

30.6	31.0	32.6	33.4	34.1

(2 points)

Use linear approximations at $(3,3,1)$:

$$L(x,y,z) = g(3,3,1) + g_x(3,3,1)(x-3) + g_y(3,3,1)(y-3) + g_z(3,3,1)(z-1)$$

$$= 30 + 6(x-3) + 4(y-3) + 5(z-1)$$

Thus:

$$g(3.5,3.1,1) \approx L(3.5,3.1,1) = 30 + 6(0.5) + 4(0.1) + 5(0)$$

$$= 30 + 3 + \frac{2}{5}$$

$$= \frac{150+15+2}{5} = \frac{167}{5} \approx 33.4$$

① Apply the right hand rule, with your thumb along v and your index along w, your middle finger should point along b. The magnitude follows from the fact that $|b| = 1$ and :

$$|v \times w| = |v|\,|w| \sin\left(\frac{\pi}{2}\right) = 1 \cdot 2 \cdot 1 = 2$$

since w points directly out of the paper

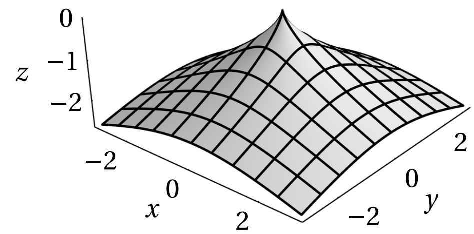
② The first two can't be solutions to Laplace's equation because the concavity of x- or y- cross sections at the maxima is negative, meaning that

$$\frac{\partial^2 u}{\partial x^2} < 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} < 0 \quad \text{so} \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \neq 0 .$$

(A similar conclusion would be reached if minima were obtained). This leaves only the last one; but also notice that this is essentially the graph of $f(x,y) = x^2 - y^2$; which has $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2 - 2 = 0$.

③ There are other ways to approach $(0,1)$ besides lines. These other paths of approach could potentially yield different limiting values. However, if the limit exists, then the limit should be the same regardless of the path chosen. In particular, it must be 3 in that case.

5. Let S be the surface $x^2+y^2=-z^3$ shown at right. Let f be a function on $\mathbb{R}^3$ with continous partial derivatives such that

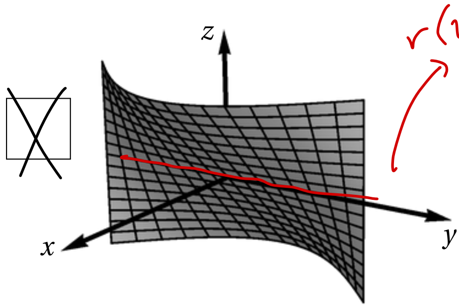


$$\nabla f(0,0,0)=\langle 1,1,3\rangle \qquad \nabla f(2,2,-2)=\langle 1,1,3\rangle$$
$$\nabla f(-2,-2,-2)=\langle 1,1,3\rangle \qquad \nabla f(2,-2,-2)=\langle 0,0,0\rangle$$

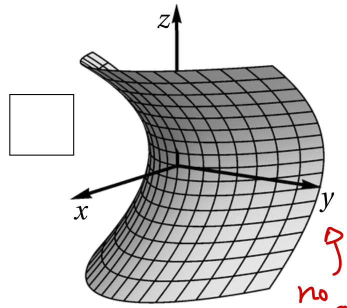
Circle every point below that can **not** be the point at which f achieves its minimum value on S. **(4 points)**

$(0,0,0)$	$(2,2,-2)$	$(-2,-2,-2)$	$(2,-2,-2)$

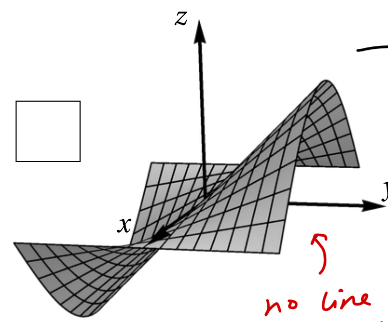
6. Let S be the surface parametrized by $\mathbf{r}(u,v)=\langle v\cos u,\ u,\ v\rangle$ for $-\pi/2\le u\le\pi/2$ and $-1\le v\le 1$. Check the box next to the picture of S below: **(2 points)**

☒ 

$r(u,0)$

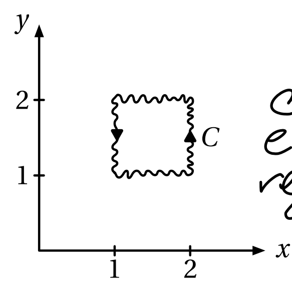
☐ 

no line on y-axis

☐ 

no line on y-axis

7. Let $\mathbf{F}(x,y)=\left\langle e^{y^2},\ 3x+2xye^{y^2}\right\rangle$, and let C be the oriented curve at right. Estimate the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(2 points)**



call the enclosed region R

−9	−6	−3	0	3	6	9

Scratch Space

(5) Set $g=x^2+y^2+z^3$

min could occur where $\nabla f=\lambda\nabla g$, $\lambda\neq 0$, by Lagrange's multiplier method or $\nabla g=\vec{0}$

$\nabla g=(2x,\ 2y,\ 3z^2)$

$\nabla g(0,0,0)=(0,0,0)$ ✓ candidate

$\nabla g(-2,-2,-2)=(-4,-4,12)$ not a nonzero multiple of $\nabla f(-2,-2,-2)$

$\nabla g(2,2,-2)=(4,4,12)$ ✓ candidate since multiple of $\nabla f(2,2,-2)$

$\nabla g(2,-2,-2)=(4,-4,12)$ could happen since $\nabla f=\vec{0}$ here

(6) Note there's a line on the y-axis!

$v=0$: $r(u,0)=(0,u,0)$

only one surface has this

(7) Use Green's:

$Q_x-P_y=3+2ye^{y^2}-2ye^{y^2}=3$

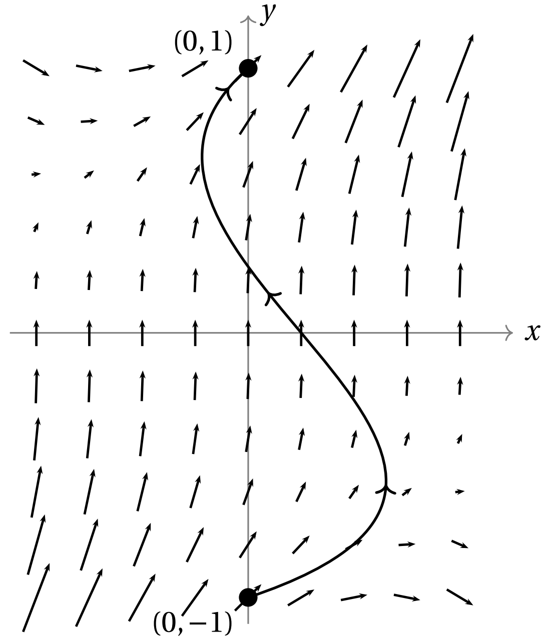
so $\int_C F\cdot d\vec{r}$

$=\iint_R 3\,dA=3\,\text{Area}\,R$

$\approx 3(1)$

$=3$

8. A vector field $\mathbf{F}$ is shown at right; for scale, here $\mathbf{F}(0,0) = \langle 0,\ 0.1\rangle$. Assuming that $\mathbf{F}$ is conservative, circle the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve shown from $(0,-1)$ to $(0,1)$.



−0.3	−0.2	−0.1	0	0.1	0.2	0.3

(2 points)

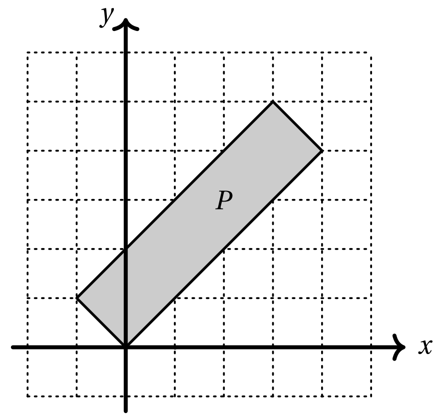
F conservative ⇒ work is path independent
take straight line path on y-axis:

$\begin{cases} x(t) = 0 \\ y(t) = t \end{cases}$ $-1 \le t \le 1$.

If $F = (P, Q)$ then Q is basically constant on y-axis with $Q \approx 0.1$ (from picture). Thus:

$\int_C F\cdot d\vec{r} = \int_{-1}^{1} Q(0,t)\, y'(t)\,dt \approx \int_{-1}^{1} 0.1\,dt = 0.2$.

9. Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(u,v) = (2u - v,\ 2u + v)$. Let P be the rectangle shown below in the (x,y)–plane, drawn against a unit-square grid. Check the box next to the region D in the (u,v)–plane below that is mapped to P by T. **(2 points)**



- [ ] $D = \{0 \le u \le 1 \text{ and } 0 \le v \le 1\}$
- [x] $D = \{0 \le u \le 2 \text{ and } 0 \le v \le 1\}$
- [ ] $D = \{0 \le u \le 1 \text{ and } 0 \le v \le 4\}$
- [ ] $D = \{0 \le u \le 4 \text{ and } 0 \le v \le 1\}$
- [ ] $D = \{-1 \le u \le 0 \text{ and } 0 \le v \le 1\}$
- [ ] $D = \{0 \le u \le 1 \text{ and } 0 \le v \le 2\}$

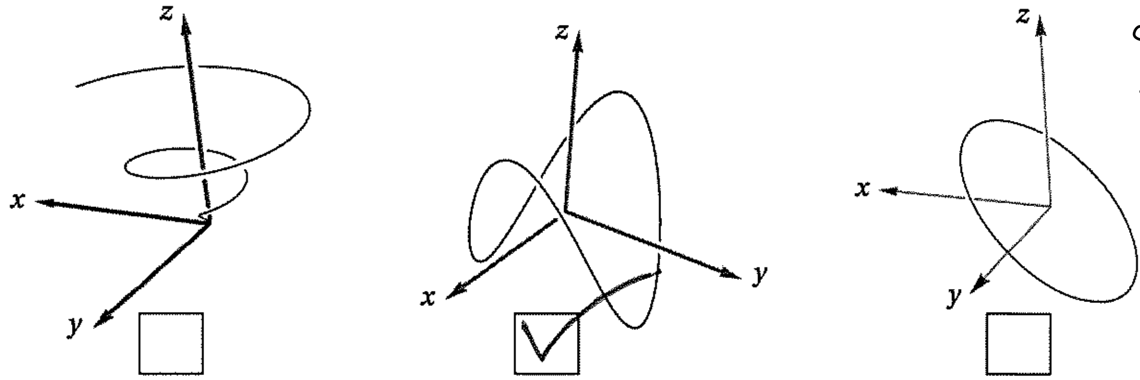
Scratch Space

(9) T is linear transformation of the plane, so D must be a parallelogram and the spanning vectors of D must be mapped by T to the spanning vectors of P, which are (4,4) and (−1,1).
Note that $T(1,0) = (2,2)$ and $T(0,1) = (-1,1)$ (use the matrix for speed)
Since T is linear it sends multiples to multiples,
so $T(2,0) = 2T(1,0) = 2(2,2) = (4,4)$. Hence, D is spanned by (2,0) and (0,1). That is the rectangle we marked.

10. Consider the curve C parametrized by $\mathbf{r}(t) = \left(\cos t, \sin t, \cos 2t\right)$, where $0 \le t \le 2\pi$.

The projection to xy plane is circle $(\cos t, \sin t)$ so rule out 1st option.
The z component changes sign four times in domain. Hence, rule out 3rd option.

(a) Check the box next to the correct sketch of C. **(2 points)**



(b) Find the work done if a particle travels along path $\mathbf{r}(t)$ under the force field given by $\mathbf{F}(x, y, z) = \langle -2y, 2x, 0\rangle$. **(4 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle -2\sin t, 2\cos t, 0\rangle \cdot \underbrace{\langle -\sin t, \cos t, -2\sin 2t\rangle}_{\vec{r}\,'(t)} dt$$

$$= \int_0^{2\pi} 2\underbrace{(\sin^2 t + \cos^2 t)}_{1}\, dt$$

$$= 4\pi$$

Total work done = 4π

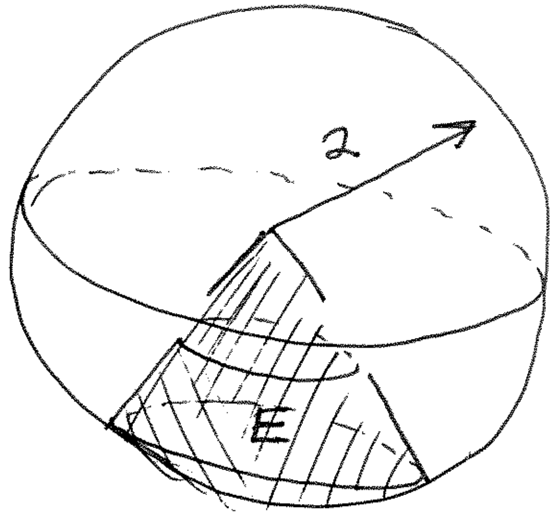
11. Let $S = \{u^2 + v^2 + w^2 = 1\}$ be the unit sphere around the origin, and let $E = \{4x^2 + (y-1)^2 + (z-3)^2 = 1\}$, which is an ellipsoid with center $(0, 1, 3)$. Find a transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(S) = E$. **(3 points)**

Need to scale 1st coor down by 1/2, then shift in the other two coor.

$$T(u, v, w) = \langle u/2, v+1, w+3 \rangle$$

12. Find the volume of the solid that lies below the cone $z = -\sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=4$. **(5 points)**

$$\mathrm{Vol}(E) = \int_0^2 \int_{\frac{3}{4}\pi}^{\pi} \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

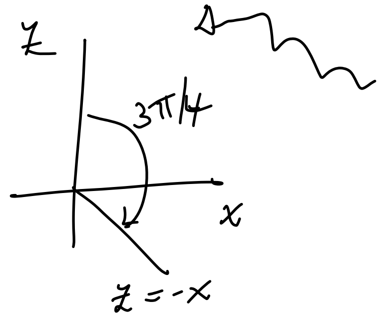


$$= \int_0^2 \rho^2 d\rho \int_0^{2\pi} d\theta \int_{3/4\pi}^{\pi} \sin\phi \, d\phi$$

$$= \left(\frac{\rho^3}{3} \Big|_0^2 \right) \cdot 2\pi \cdot \left(-\cos\phi \Big|_{3/4\pi}^{\pi} \right)$$

$$= \frac{8}{3} \cdot 2\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$

To find the ϕ bounds, look at xz plane in just one quadrant (say the 4^{th} quadrant where $x \geq 0$). There the eq. $z = -\sqrt{x^2+y^2}$ becomes $z = -x$ and this makes an angle $3\pi/4$ w/ z-axis starting at the top.



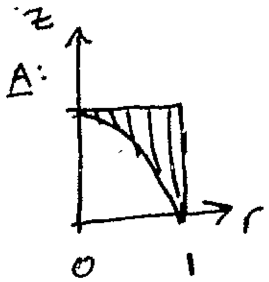
Write your volume integral here:

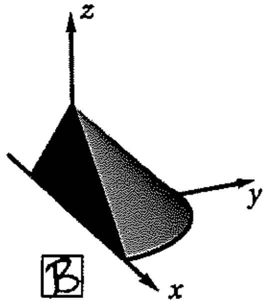
$$\boxed{\int_0^2 \int_{3/4\pi}^{\pi} \int_0^{2\pi} \rho^2 \sin\phi \quad d\theta \, d\phi \, d\rho}$$

Your final answer: Volume = $\frac{16\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$

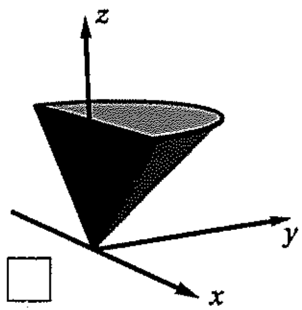
13. For each of the integrals: **(A)** $\int_0^{\pi}\int_0^1\int_{1-r^2}^1 f(r,\theta,z)\,r\,dz\,dr\,d\theta$ and **(B)** $\int_0^{\pi}\int_0^1\int_0^{1-z} f(r,\theta,z)\,r\,dr\,dz\,d\theta$ label the solid corresponding to the region of integration below. **(2 points each)**

A:

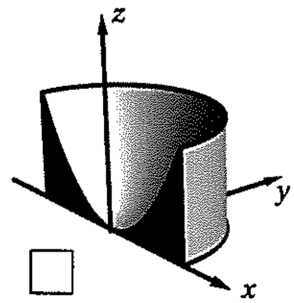




$\boxed{B}$

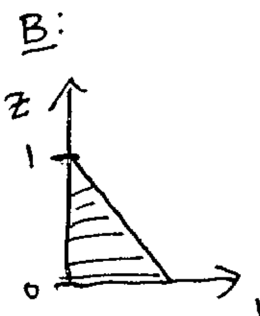


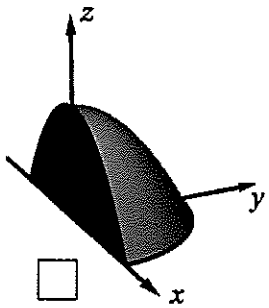
$\boxed{}$



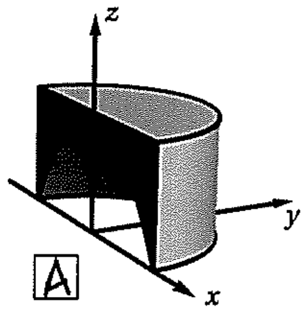
$\boxed{}$

B:

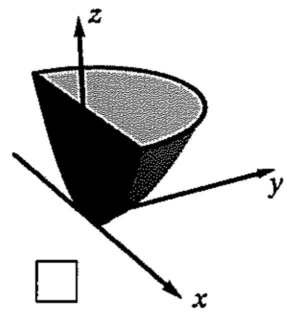




$\boxed{}$



$\boxed{A}$

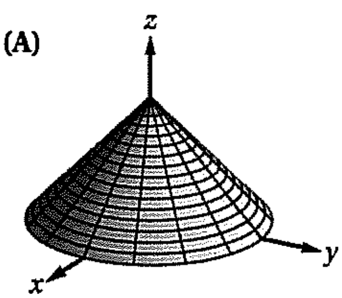


$\boxed{}$

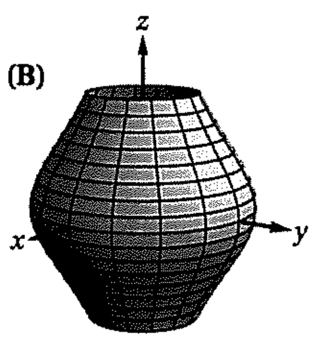
14. Label the boxes next to the parametrizations that correspond to the following two surfaces: **(2 points each)**

(explanations on next page)

(A)



(B)



$\boxed{B}$ $\mathbf{r}(u,v) = \left\langle \dfrac{\cos v}{1+u^2}, \dfrac{\sin v}{1+u^2}, u \right\rangle$ for $-1 \le u \le 1, 0 \le v \le 2\pi$

$\boxed{}$ $\mathbf{r}(u,v) = \langle \cos v, \sin v, u\rangle$ for $-1 \le u \le 1, 0 \le v \le 2\pi$

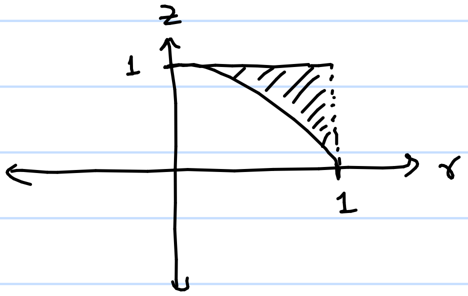
$\boxed{A}$ $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 1-u\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$

$\boxed{}$ $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 1-u^2\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$

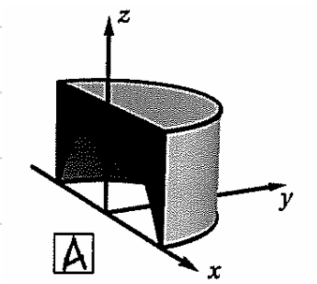
$\boxed{}$ $\mathbf{r}(u,v) = \langle \sin u\cos v, \sin u\sin v, \cos u\rangle$ for $0 \le u \le \pi/2, 0 \le v \le 2\pi$

13 (a) $\displaystyle\int_0^{\pi}\int_0^{1}\int_{1-r^2}^{1} f(r,\theta,z)\, r\, dz\, dr\, d\theta$

The cross-section along rz-plane is given by the bounds $0 \le r \le 1$, $1-r^2 \le z \le 1$ for any θ



The only option with this cross-section is



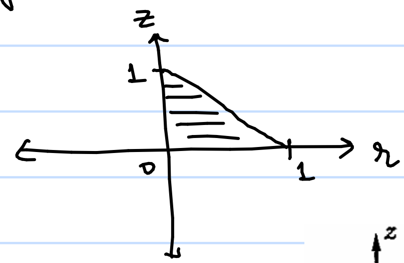
$\boxed{A}$

(b) $\displaystyle\underbrace{\int_0^{\pi}\int_0^{1}\int_0^{1-z}}\, f(r,\theta,z)\, r\, dr\, dz\, d\theta$

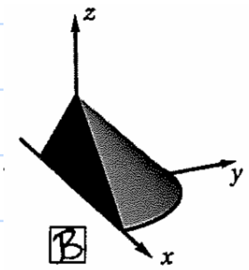
This tells us that the cross-section along rz-plane (for any θ) should be given by

$0 \le z \le 1$

$0 \le r \le 1-z$



The only option whose intersection with vertical planes looks like above is



$\boxed{B}$

14 (i) $\mathbf{r}(u,v) = \left\langle \dfrac{\cos v}{1+u^2}, \dfrac{\sin v}{1+u^2}, u \right\rangle$ for $-1 \le u \le 1, 0 \le v \le 2\pi$ $\rightarrow x^2+y^2 = \left(\dfrac{1}{1+u^2}\right)^2 = \left(\dfrac{1}{1+z^2}\right)^2$

(ii) $\mathbf{r}(u,v) = \langle \cos v, \sin v, u\rangle$ for $-1 \le u \le 1, 0 \le v \le 2\pi$ $\rightarrow x^2+y^2 = u^2 = z^2$

(iii) $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 1-u\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$ $\rightarrow x^2+y^2 = u^2 = (1-z)^2$

(iv) $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 1-u^2\rangle$ for $0 \le u \le 1, 0 \le v \le 2\pi$ $\rightarrow x^2+y^2 = u^2 = 1-z$

(v) $\mathbf{r}(u,v) = \langle \sin u\cos v, \sin u \sin v, \cos u\rangle$ for $0 \le u \le \pi/2, 0 \le v \le 2\pi$ $\rightarrow x^2+y^2 = \sin^2 u = 1-\cos^2 u = 1-z^2$

Continued on next page →

(A) The level sets are circles and the radius of circles decreases at a constant rate as we move up.
So, we want a surface whose radius decreases constantly with z

For

$$r(u,v) = (u\cos v,\ u\sin v,\ 1-u)$$

radius at height $z = 1-z$

Thus this the Correct choice.

(B) We want a surface whose level sets are circles whose radius first increases at a non-constant rate and then decreases.

For $r(u,v) = \left(\dfrac{\cos v}{1+u^2},\ \dfrac{\sin v}{1+u^2},\ u\right)$

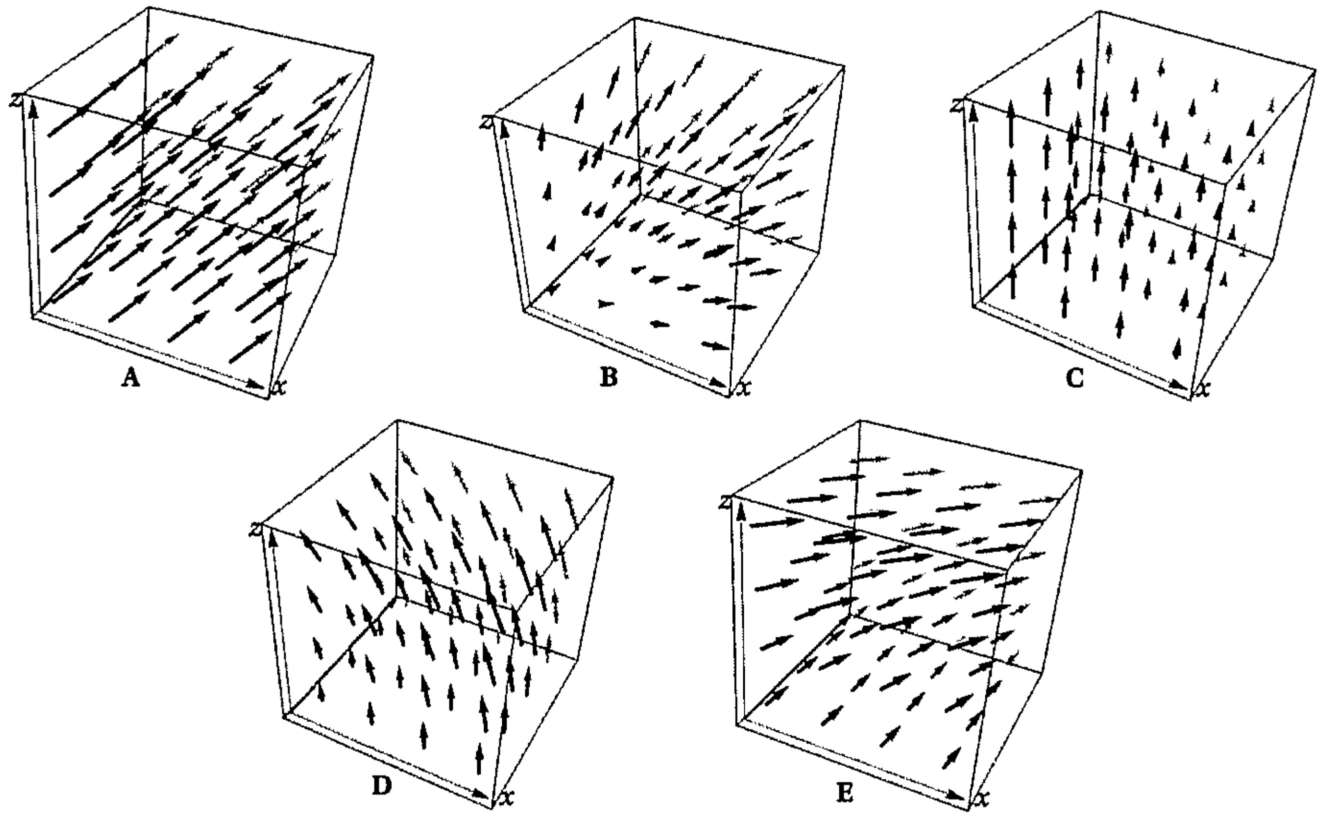
radius at ht $z = \dfrac{1}{1+z^2}$

Thus, for $-1 \le z \le 1$, radius increases uptil $z=0$ and then decreases.

In all the remaining options this does not happen.

■

15. Here are plots of five vector fields on the box where $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. **(2 points each)**

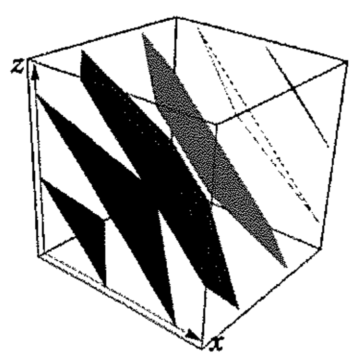


(a) Circle the name of the vector field that is given by $\langle -z,\, 0,\, 1+x \rangle$: | A B C D E |

(b) Exactly one of these vector fields has nonzero divergence. Circle it: | A B C D E |

(c) The vector field **C** is conservative: | true false |

(d) Which vector field is the gradient of a function f whose level sets are shown below? | A B C D E |



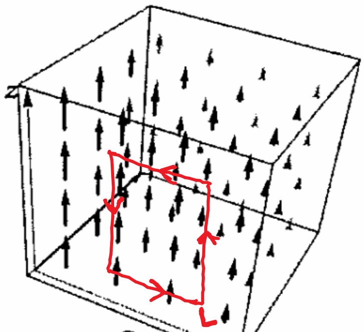
(explanation on next page)

15 (a) $\langle -z, 0, 1+x \rangle$ has no y-component. So, its either C or D.
Since the x-coordinate is $-z$, the arrows should point towards the left. Thus D is the correct choice.

(b) Imagine that the arrows tell us the velocity of a moving fluid. Then the divergence is non-zero if the fluid is expanding or contracting.
It is clear that in B, the fluid is expanding.

(c) Consider the following closed loop L in the xz-plane

The contributions from the top & bottom edge cancel each other.
However, the contribution of the left vertical edge is greater than the one on the right.



C

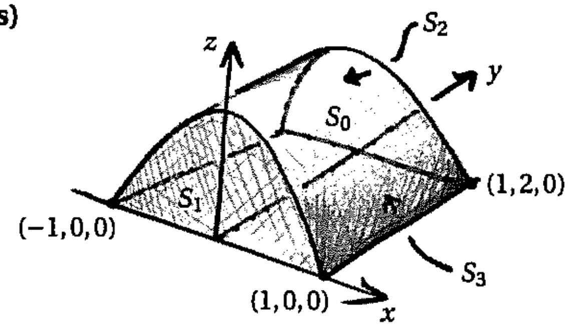
$$\text{so} \int_L C \cdot dr \neq 0$$

Hence, C is not conservative

(d). The gradient vector field has to be perpendicular to the level sets.
Thus, A is the correct choice.

16. Let E be the solid region shown below, where ∂E is decomposed into the four subsurfaces S_i indicated; here the top S_0 is where $z + x^2 = 1$, the front S_1 is in the xz-plane, the back is S_2, and the bottom is S_3.

(a) Use a triple integral to compute the volume of E. **(~~5~~ points)** 4



$$\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1-x^2} 1 \, dz\, dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{2} 1 - x^2 \, dy\, dx = \int_{-1}^{1} 2 - 2x^2 \, dx$$

$$= 2x - \frac{2}{3}x^3 \Big|_{x=-1}^{1} = 8/3$$

$\boxed{\text{Vol}(E) = 8/3}$

(b) Give a parameterization of S_0 and use it to directly compute the flux of $\mathbf{F} = \langle 1, 0, z+2 \rangle$ through S_0 with respect to the upwards normals. **(5 points)**

$\vec{r}(u,v) = \langle u, v, 1-u^2 \rangle \quad D = \{ -1 \le u \le 1 \text{ and } 0 \le v \le 2 \}$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & 0 \end{vmatrix} = \langle 2u, 0, 1 \rangle$$

$$\text{Flux} = \int_0^2 \int_{-1}^{1} \underbrace{\langle 1, 0, 3-u^2 \rangle}_{\vec{F}(\vec{r}(u,v))} \cdot \overbrace{\langle 2u, 0, 1 \rangle \, du\, dv}^{\vec{r}_u \times \vec{r}_v \, du\, dv}$$

$$= \int_0^2 \int_{-1}^{1} 2u + 3 - u^2 \, du\, dv = \int_0^2 u^2 + 3u - \frac{1}{3}u^3 \Big|_{u=-1}^{u=1} dv$$

$$= \int_0^2 6 - 2/3 \, dv = 2 \cdot 16/3$$

$\boxed{\iint_{S_0} \mathbf{F} \cdot d\mathbf{S} = 32/3}$

(c) The flux of $\mathbf{F}$ through exactly two of S_1, S_2, and S_3 is zero. Circle the one where the flux is **nonzero**: **(1 point)**

S_1	S_2	S_3

Since $\vec{F}$ has no y-component & the normal $\vec{n}$ to S_1 and S_2 points along y axis, the flux through S_1 and S_2 is zero.

17. Consider the vector field $\mathbf{F} = \langle -y,\ x+z,\ x^2+z \rangle$ on $\mathbb{R}^3$.

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+z & x^2+z \end{vmatrix}$$

$$= \langle -1, -2x, 2 \rangle$$

(a) Circle the curl of $\mathbf{F}$: **(2 points)**

curl $\mathbf{F}$ =	$\langle z, -y, x\rangle$	$\langle -1, 2x, 2\rangle$	$\langle 0, 1, 2x\rangle$	$\langle -1, -2x, 2\rangle$	$\langle -y, 2x, 2z\rangle$

(b) Suppose C is a closed curve in the plane P given by $x - z = 1$. Assuming C bounds a region R of area 10 in P, determine the absolute value of $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(4 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl}\,\vec{F})\cdot \vec{n}\, dA = \iint_R \langle -1, -2x, 2\rangle \cdot \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle\, dA$$

Stokes' Theorem

$$\iint_R \frac{1}{\sqrt{2}}(-1-2)\, dA = -\frac{3}{\sqrt{2}}\,\text{Area}(R) = \frac{-30}{\sqrt{2}}$$

Normal to R comes from eqn for plane: $x - z = 1$

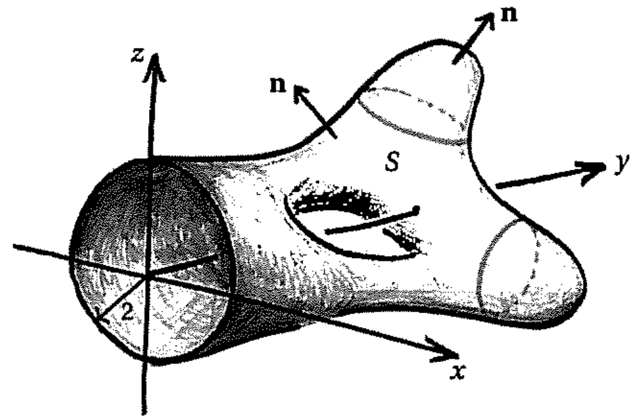
$\rightsquigarrow \langle 1, 0, -1\rangle \xrightarrow{\text{make unit}} \vec{n} = \frac{1}{\sqrt{2}}\langle 1, 0, -1\rangle$

$$\boxed{\left|\int_C \mathbf{F}\cdot d\mathbf{r}\right| = 30/\sqrt{2}}$$

Scratch Space

18. The surface S shown below has boundary the circle of radius 2 in the xz-plane. With respect to the normal vector field indicated, compute the flux of $\mathbf{G} = \langle 0, 3, 0 \rangle$ through S. **(5 points)**

Backup question: If you can't find the exact answer, determine whether the flux is positive, zero, or negative and write that in the answer box for partial credit.



Let D be the disc of radius 2 in the xz-plane.

Then D and S together bound a solid region E. Then

$$0 = \underbrace{\iiint_E \operatorname{div} \vec{G} = \iint_{\partial E} \vec{G} \cdot \vec{n}\, dA}_{\text{Divergence Theorem}} = \iint_D \vec{G} \cdot \vec{n}\, dA + \iint_S \vec{G} \cdot \vec{n}\, dA.$$

Thus $\iint_S G \cdot \vec{n}\, dA = -\iint_D \vec{G} \cdot \vec{n}\, dA$ ← outwards normal for E $= -\vec{j}$

$$= \iint_D \vec{G} \cdot \vec{j}\, dA$$

$$= \iint_D 3\, dA = 3\, \text{Area}(D) = 3\pi 2^2 = 12\pi$$

$\boxed{\iint_S \mathbf{G} \cdot d\mathbf{S} = 12\pi}$

Scratch Space

19. Suppose $g(x,y,z) = e^x + y\cos z$ and $\mathbf{G} = \nabla g = \langle e^x, \cos z, -y\sin z\rangle$ is its gradient vector field. **(2 points each)**

(a) Let C denote the parametric curve $\mathbf{r}(t) = \langle 2,\ t,\ \pi t\rangle$ for $0 \le t \le 1$. The integral $\int_C \mathbf{G}\cdot d\mathbf{r}$ is:

e^2	π	-1	0	1	e^2-1

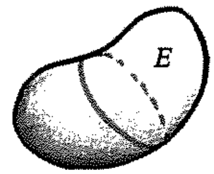
(b) Let S denote the hemisphere defined by $x^2+y^2+z^2=1$ and $z \ge 0$; let $\mathbf{n}$ denote the upward unit normal. The integral $\iint_S \operatorname{curl}\mathbf{G}\cdot\mathbf{n}\,dA$ is:

positive	zero	negative

(c) Consider the vector field $\mathbf{F} = \langle -y,\ x+z,\ y\rangle$. The vector field $\mathbf{F}$ is:

conservative	not conservative

20. Suppose E is a solid region in $\mathbb{R}^3$ looking like the picture at right.



(a) Suppose E has volume 10 and is made of material of constant density. Check the box next to the integral that must compute the x-coordinate of its center of mass. **(2 points)** (explanation on next page)

☐ $\frac{1}{30}\iint_{\partial E} \mathbf{A}\cdot d\mathbf{S}$ for $\mathbf{A} = \langle x,\ y,\ z\rangle$ ☐ $\iiint_E x\,dV$ ☐ $\frac{1}{10}\iiint_E y\,dV$

☑ $\frac{1}{20}\iint_{\partial E} \mathbf{B}\cdot d\mathbf{S}$ for $\mathbf{B} = \langle z,\ xy,\ xz\rangle$ ☐ None of these.

(b) Assuming the origin lies inside of E, determine the flux of $\mathbf{H} = \dfrac{-3}{(x^2+y^2+z^2)^{3/2}}\langle x,\ y,\ z\rangle$ through ∂E. **(2 points)**

Use Gauss's Law:

$\iint_{\partial E} \mathbf{H}\cdot d\mathbf{S} = \boxed{-12\pi}$

(explanation on next page)

Scratch Space

19(a) $\vec{r}(0) = \langle 2,0,0\rangle$ $\vec{r}(1) = \langle 2,1,\pi\rangle$

$g(\vec{r}(0)) = e^2$ $g(\vec{r}(1)) = e^2-1$

Fund Thm $\Rightarrow \int_C \vec{G}\cdot d\vec{r} = e^2-1-(e^2) = -1$

19(b) $\operatorname{curl}\vec{G} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & \cos z & -y\sin z \end{vmatrix} = \langle -\sin z+\sin z, 0, 0\rangle = \vec{0}$

c) $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x+z & y \end{vmatrix} = \langle 0,0,2\rangle \Rightarrow$ vector field is not conservative since the curl is nonzero

20. (a)

$$x\text{-coordinate of center of mass} = \frac{1}{\text{volume}} \iiint_E x \, dV$$

For $B = \langle z, xy, xz \rangle$,

$$\operatorname{div} B = \frac{\partial z}{\partial x} + \frac{\partial (xy)}{\partial y} + \frac{\partial (xz)}{\partial z} = x + x = 2x$$

$$\therefore \iint_{\partial E} B \cdot dS \underset{\text{Divergence Thm}}{=} \iiint_E \operatorname{div} B \, dV$$

$$= \iiint_E 2x \, dV$$

$$\Rightarrow \frac{1}{20} \iint_{\partial E} B \cdot dS = \frac{1}{20} \iiint_E 2x \, dV = \frac{1}{10} \iiint_E x \, dV$$

(recall, volume of E = 10)

(b)

$$H = \frac{-3}{(x^2+y^2+z^2)^{3/2}} \langle x, y, z \rangle$$

$$= \frac{-3 \cdot (4\pi\varepsilon_0)}{4\pi\varepsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \quad \left(\text{where } \vec{r} = \langle x, y, z \rangle\right)$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \quad \left(\text{where } q = -12\pi\varepsilon_0\right)$$

By Gauss's law,

$$\iint_{\partial E} H \cdot dS = \iint_{\partial E} \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \, dS = \frac{q}{\varepsilon_0} = \frac{-12\pi\varepsilon_0}{\varepsilon_0} = -12\pi$$

■