1. Consider the three points $A = (0,1,0)$, $B = (1,1,1)$, and $C = (0,2,1)$ in $\mathbb{R}^3$. For each part, circle the best answer. **(1 point each)**

(a) The projection of the vector $\overrightarrow{AC}$ onto $\overrightarrow{AB}$ is: $\langle 0, 2, 2\rangle$ $\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$ $\left\langle \frac{1}{2}, 0, \frac{1}{2}\right\rangle$ $\left\langle \frac{1}{2}, \frac{1}{2}, 0\right\rangle$

(b) The area of the triangle formed by these three points is: $\sqrt{3}$ $\frac{\sqrt{3}}{2}$ 3 $\frac{3}{2}$

(c) $\overrightarrow{AB} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) =$ -3 -2 -1 0 1 2 3

(d) For the vector $\mathbf{v} = \langle 0, 2, 2\rangle$, the angle between $\overrightarrow{AC}$ and $\mathbf{v}$ is: 0 $\pi/4$ $\pi/2$ π $3\pi/2$

2. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ has the table of values and partial derivatives shown at right. For $x(r,s) = 2r - s$ and $y(r,s) = s^2 - 4r$, let $F(r,s) = f(x(r,s), y(r,s))$ be their composition with f.
Circle the value of $\frac{\partial F}{\partial r}(1,2)$: 24 -24 40 -40 -11 11
(2 points)

(x,y)	$f(x,y)$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
(0,0)	3	4	8
(1,2)	6	2	5
(3,3)	19	−8	5
(4,3)	7	3	2

Scratch Space

1. a) $\vec{b} = \overrightarrow{AB} = \langle 1, 0, 1\rangle$ $\quad \vec{c} = \overrightarrow{AC} = \langle 0, 1, 1\rangle$

$$\operatorname{proj}_{\vec{b}} \vec{c} = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}|^2}\vec{b} = \frac{1}{2}\langle 1, 0, 1\rangle = \langle 1/2, 0, 1/2\rangle$$

b) $\vec{b}\times\vec{c} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \langle -1, -1, 1\rangle$ so area $=$ $\frac{1}{2}|\vec{b}\times\vec{c}| = \sqrt{3}/2$.

c) $\vec{b}$ is $\perp$ to $\vec{b}\times$ (anything).

d) $\vec{v}$ and $\vec{c}$ point in the same direction.

2. $$\frac{\partial F}{\partial r}(1,2) = \underbrace{\frac{\partial F}{\partial x}(\underset{0}{x(1,2)}, \underset{0}{y(1,2)})}_{4} \cdot \underset{2}{\frac{\partial x}{\partial r}(1,2)} + \underbrace{\frac{\partial F}{\partial y}(x(1,2), y(1,2))}_{8} \cdot \underset{-4}{\frac{\partial y}{\partial r}}$$

$$= 4\cdot 2 + 8\cdot(-4) = -24.$$

3. **Exactly one** of the limits at right exists.

Circle the limit **that exists:** A B

Justify your answer either by showing that the limit you circled exists and computing its value, or by showing that the other limit does not exist. **(3 points)**

(A) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}$

(B) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+x}{\sqrt{x^2+y^2}}$

(A) exists: In polar, the fn is $\frac{\sin r}{r}$ and so the limit is equivalent to the 1-var limit $\displaystyle\lim_{r\to 0} \frac{\sin r}{r}$

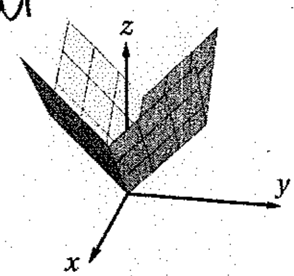
$= \displaystyle\lim_{r\to 0} \frac{\cos r}{1} = 1$ by L'Hopital's Rule.

(B) Does not exist: Again in polar, the fn is

$\dfrac{r(r\cos^2\theta + \cos\theta)}{r} = r\cos^2\theta + \cos\theta$

Thus, if we approach along a line (so θ = fixed) we get $\cos\theta$ as the limit, which can be anything in $[-1, 1]$. So the limit D.N.E.

4) When $x, y > 0$, $f = x + y$ which is differentiable at $(1,1)$. At $(0,0)$, f doesn't jump, but there isn't a good tangent plane.



4. Consider the function $f(x,y) = |x| + |y|$, whose graph is shown at right. Circle the phrase that best completes each sentence. **(1 point each)**

(a) At the point $(1,1)$, the function f is | continuous | differentiable | both continuous and differentiable |

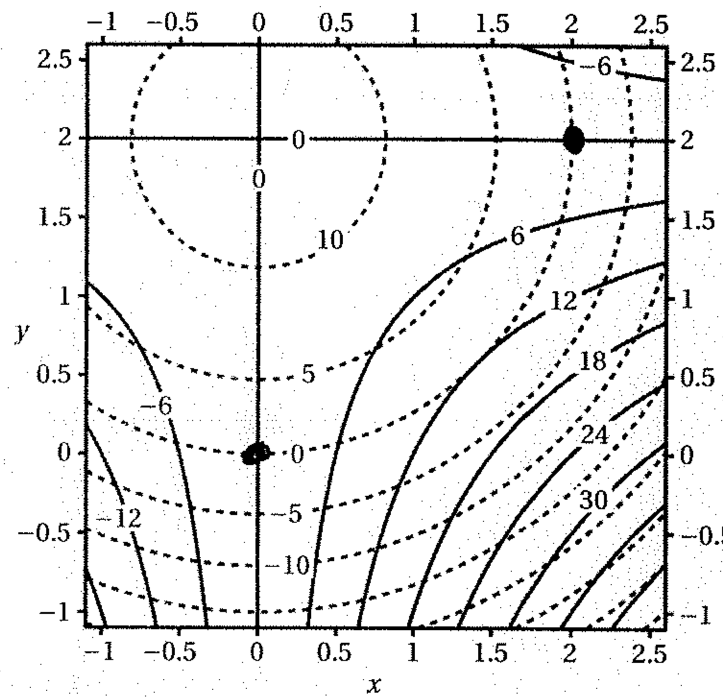
(b) At the point $(0,0)$, the function f is | continuous | differentiable | both continuous and differentiable |

5. Suppose that $g(x, y, z)$ is a function and $g(1,0,3) = 6$ and $\nabla g(1,0,3) = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$. Use linear approximation to estimate the value of $g(1.2, -0.1, 2.9)$. **(2 points)**

$g(1.2,-0.1,2.9) \approx$	5.5	5.7	5.8	5.9	6	6.1	6.2	6.3	6.4	6.5

6. The level curves of the partial derivatives f_x (solid lines) and f_y (dashed lines) of a function $f(x, y)$ are shown at right. There are **exactly two** critical points of f in the domain shown in the picture. Find both of them and classify each as a local minimum, local maximum, or saddle. **(? points)**

Critical pts are where $f_x = f_y = 0$, i.e. where the level sets $f_x = 0$ and $f_y = 0$ cross



(0 , 0)	which is a	local min	local max	saddle
(2 , 2)	which is a	local min	local max	saddle

Scratch Space

5. $g(1.2, -0.1, 2.9) \approx g(1,0,3) + g_x(1,0,3)\cdot(0.2) + g_y(-)\cdot(-0.1) + g_z(-)\cdot(-0.1)$
$= 6 + 2\cdot 0.2 + 3\cdot(-0.1) + 4\cdot(-0.1) = 5.7$

6. At $(0,0)$, have $f_{xx} > 0$, $f_{yx} = 0$, and $f_{yy} > 0$. Hence $D = \begin{vmatrix} f_{xx} & f_{yx} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$ and $f_{xx} > 0 \Rightarrow$ local min

At $(2,2)$, have $f_{xx} = 0$, $f_{yy} = 0$ and $f_{xy} < 0$ so $D < 0 \Rightarrow$ saddle.

7. Find the absolute maximum and minimum values of the function $f(x, y) = 2x^2 + y^2 + 5$ subject to the constraint $\underbrace{x^2 + y^2}_{g(x,y)} \leq 4$. **(? points)**

Lagrange Mult: $\nabla f = \langle 4x, 2y \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$

Eqns: $2x = \lambda x \quad y = \lambda y \quad x^2+y^2=4$

$x \neq 0 \Rightarrow \lambda = 2 \Rightarrow y = 2y \Rightarrow y = 0 \Rightarrow x = \pm 2$

$y \neq 0 \Rightarrow \lambda = 1 \Rightarrow 2x = x \Rightarrow x = 0 \Rightarrow y = \pm 2.$

So four crit pts: $(\pm 2, 0)$ where $f = 13$
$(0, \pm 2)$ where $f = 9$

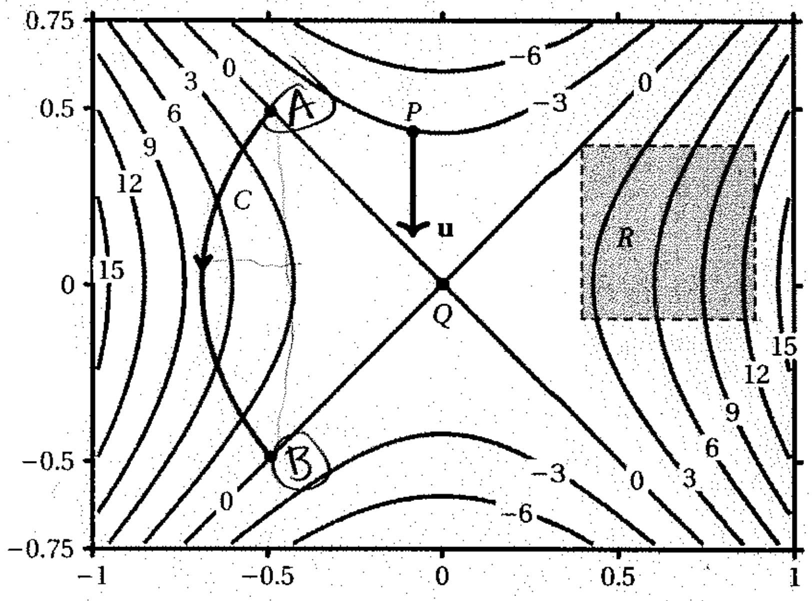
Interior Crit Pts: $\nabla f = \langle 4x, 2y \rangle = 0$, i.e.
$x = y = 0$ and $f = 5$.

The Exterme Value Thm says that since f is cont. and $D = \{x^2+y^2 \leq 4\}$ both closed and bound then f has abs min/max on D which must occur at the five pts enumerated above.

Hence:

Absolute max value =	13
Absolute min value =	5

8. The contour map of a differentiable function f is shown at right. For each part, circle the best answer. **(2 points each)**



(a) The directional derivative $D_{\mathbf{u}} f(P)$ is:

positive	negative	zero

This is because at point P, f is increasing in the direction u.

(b) Estimate $\int_C f\,ds$:

−8.1	−5.4	−2.7	0	2.7	5.4	8.1

(c) Estimate $\iint_R f\,dA$:

−4.8	−3.2	−1.6	0	1.6	3.2	4.8

(d) The point Q is:

a local maximum	a local minimum	a saddle	not a critical point

(e) Find $\int_C \nabla f \cdot d\mathbf{r}$:

−12	−9	−6	−3	0	3	6	9	12

Scratch Space

b) $\int_C f\,ds = (\text{Average of } f \text{ on } C)(\text{length of } C)$

$\approx \left(4 \text{ since min on } C \text{ is } 0 \text{ and max about } 8\right)\left(1.2 \text{ since line joining endpts has len } 1\right) = 4.8$

c) $\iint_R f\,dA = (\text{Average of } f \text{ on } C)(\text{Area } R)$

$\approx (6)\left(\tfrac{1}{2} \times \tfrac{1}{2}\right) = 6/4 = 3/2 = 1.5$

d) f increases as move horizontally and decreases as move vertically.

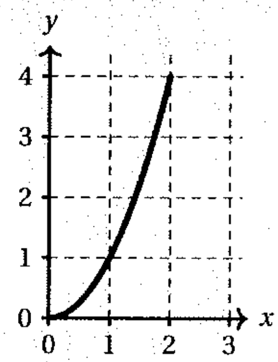
e) $\int_C \nabla f \cdot dr = f(B) - f(A) = 0 - 0 = 0$

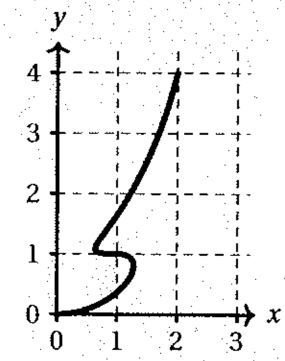
by the Fund. Thm of Line Integrals.

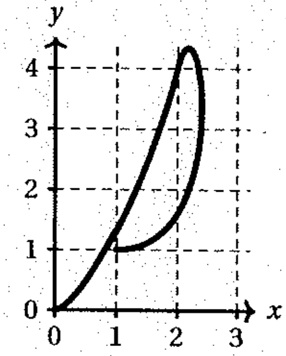
9. Suppose that $\mathbf{r}:[0,2]\to\mathbb{R}^2$ is a parametric curve in the plane and that $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the values given in the table at left. Check the box below to the picture that could be a plot of $\mathbf{r}(t)$. **(2 points)**

t	$\mathbf{r}(t)$	$\mathbf{r}'(t)$
0	$(0,0)$	$\mathbf{i}$
1	$(1,1)$	$-\mathbf{i}$
2	$(2,4)$	$\mathbf{i}+4\mathbf{j}$

The first two graphs experience correct points along the line as t=0,1,2. The second graph has the correct derivative at (1,1).





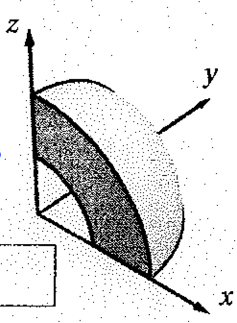


10. For each of the integrals below, label the picture of the corresponding region of integration. **(2 points each)**

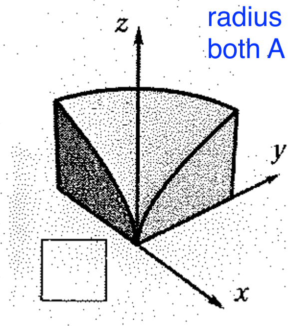
(A) $\displaystyle\int_1^2\int_0^{\pi/2}\int_{\pi/2}^{\pi} f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$

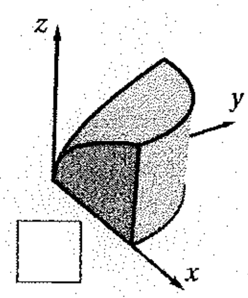
(B) $\displaystyle\int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^{\phi} g(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Wrong theta bounds for both A and B

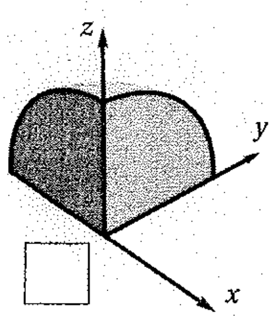


Wrong bounds of radius for both A and B

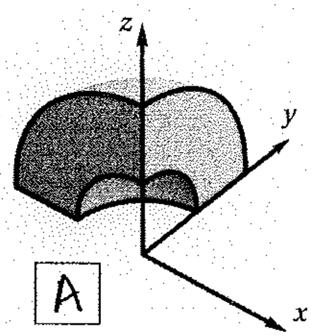




Wrong theta bounds for both A and B

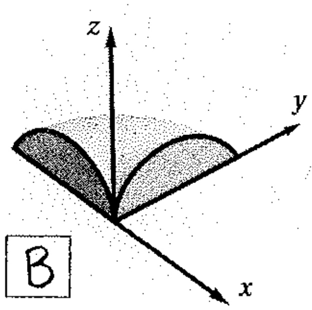


Wrong rho bounds for both A and B



A

This one has the correct theta bounds, and the radius goes from 1 to 2



B

The top surface corresponds to rho=phi

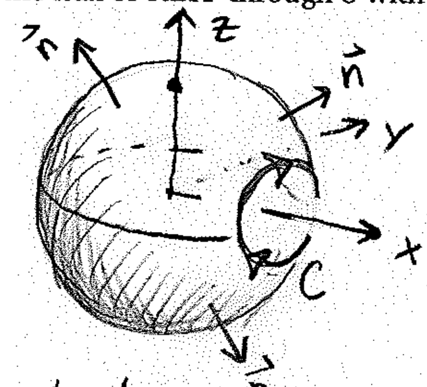
Wron

11. (a) Consider the vector field $\mathbf{F} = \langle yz, -xz, yx \rangle$ on $\mathbb{R}^3$. Compute the curl of $\mathbf{F}$. **(2 points)**

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & yx \end{vmatrix} = (x+x)\vec{i} - (y-y)\vec{j} + (-z-z)\vec{k}$$

$$\boxed{\operatorname{curl}\mathbf{F} = \langle 2x, 0, -2z \rangle}$$

(b) Let S be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \leq 3$. Find the flux of $\operatorname{curl}\mathbf{F}$ through S with respect to the outward pointing unit normal vector field. **(? points)**



By Stokes, $\iint_S (\operatorname{curl}\vec{F}) \cdot n \, dA = \int_C \vec{F} \cdot d\vec{r}$. Here C is the circle in the $x=3$ plane of radius 2 since $3^2 + y^2 + z^2 = 13 \Rightarrow y^2 + z^2 = 4$, oriented as shown. So can use $\vec{r}(t) = \langle 3, 2\sin t, 2\cos t \rangle$ for $0 \leq t \leq 2\pi$ to param. Now $\int_C F \cdot dr =$

$$\int_0^{2\pi} \underbrace{\vec{F}(\vec{r}(t))}_{\langle 4\sin t\cos t,\, -6\cos t,\, 6\sin t\rangle} \cdot \underbrace{\vec{r}\,'(t)}_{\langle 0,\, 2\cos t,\, -2\sin t\rangle} \, dt = \int_0^{2\pi} -12\cos^2 t - 12\sin^2 t \, dt = \int_0^{2\pi} -12 \, dt = -24\pi$$

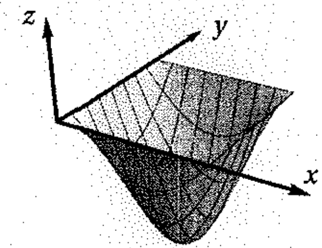
$$\boxed{\text{flux} = -24\pi}$$

(c) Let D be the of portion of sphere $x^2 + y^2 + z^2 = 13$ where $x \geq 3$. Find the flux of $\operatorname{curl}\mathbf{F}$ through D with respect to the outward pointing unit normal vector field. **(2 points)**

For D, get the other orient on C, so ans. if − what we had in (a)

$$\boxed{\text{flux} = +24\pi}$$

12. (a) Let S be the portion of the surface $z = -\sin(x)\sin(y)$ where $0 \le x \le \pi$ and $0 \le y \le \pi$ which is shown at right. Use a parameterization to find the flux of $\mathbf{F} = \langle 0, 0, 2z+1\rangle$ through S with respect to the downward normals. **(? points)**



$$\vec{r}(u,v) = \langle u, v, -\sin u \sin v\rangle \qquad 0 \le u, v \le \pi.$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -\cos u \sin v \\ 0 & 1 & -\sin u \cos v \end{vmatrix} = \underbrace{\langle \cos u \sin v, \sin u \cos v, 1\rangle}_{\text{upwards, so use } - \text{ this}}$$

$$\iint_S \vec{F}\cdot\vec{n}\, dA = \int_0^\pi\int_0^\pi \langle 0,0,-2\sin u \sin v + 1\rangle \cdot \underbrace{\langle *, *, -1\rangle}_{-\vec{r}_u\times\vec{r}_v} du\,dv$$

$$= \int_0^\pi\int_0^\pi 2\sin u \sin v - 1\, du\,dv$$

$$= 2\underbrace{\int_0^\pi \sin u\, du}\int_0^\pi \sin v\, dv - \int_0^\pi\int_0^\pi 1\, du\,dv = 2\cdot 2\cdot 2 - \pi^2$$

$$= -\cos u \Big|_{u=0}^{u=\pi} = 2$$

flux = $\boxed{8 - \pi^2}$

(b) Let E be the region below the xy-plane and above S. Use an integral theorem to compute the flux of $\mathbf{F}$ through ∂E with respect to the outward normals. **(? points)**

Divergence Thm: $\displaystyle\iint_{\partial E} \vec{F}\cdot\vec{n}\, dA = \iiint_E \underbrace{\operatorname{div}\vec{F}}_{=2}\, dV$

$$= \int_0^\pi\int_0^\pi\int_{-\sin x \sin y}^0 2\, dz\,dy\,dx = \int_0^\pi\int_0^\pi 2\sin x \sin y\, dy\,dx$$

$$= 2\int_0^\pi \sin x\, dx \int_0^\pi \sin y\, dy = 8$$

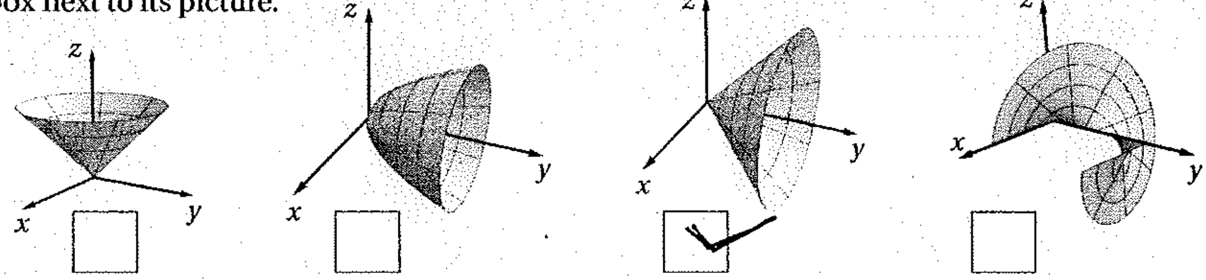
flux = $\boxed{8}$

(c) Your answers in (b) and (c) should differ. Explain what accounts for the difference. **(1 point)**

∂E consists of both S and the square $R = \{0 \le x, y \le \pi\}$ in the xy plane. The flux through R is

$$\iint_R \vec{F}\cdot\vec{n}\, dA = \iint_R \langle 0,0,1\rangle\cdot\langle 0,0,1\rangle\, dA = \text{Area}(R) = \pi^2$$

13. (a) Let S be the surface parameterized by $\mathbf{r}(u,v) = \langle v\cos u,\ v,\ v\sin u\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 2$. Mark the box next to its picture. **(2 points)**



see explanation on following page

(b) Use the parameterization to find the tangent plane to S at $(1, \sqrt{2}, 1)$. **(? points)**

This pt has $v = y = \sqrt{2}$ and $x = 1 = \sqrt{2}\cos u \Rightarrow u = \pi/4$.

Now $\vec{r}_u \times \vec{r}_v$ is $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & 0 & v\cos u \\ \cos u & 1 & \sin u \end{vmatrix} = \langle -v\cos u, v, -v\sin u\rangle$

which is $\langle -1, \sqrt{2}, -1\rangle$

when $u = \pi/4$ and $v = \sqrt{2}$. Hence the tangent plane is

$-1(x-1) + \sqrt{2}(y - \sqrt{2}) - 1\cdot(z-1) = 0$ or

equivalently

Equation: $\boxed{-1}\,x + \boxed{\sqrt{2}}\,y + \boxed{-1}\,z = \boxed{0}$

(c) The surface S has area $4\sqrt{2}\pi$. Find the average of $f(x,y,z) = y$ on S. **(? points)**

$$\iint_S f\,dA = \int_0^{2\pi}\int_0^2 \underbrace{v}_{f(\vec{r}(u,v))}\ \underbrace{|\vec{r}_u \times \vec{r}_v|}_{\sqrt{v^2\cos^2 u + v^2 + v^2\sin^2 u}}\,du\,dv = \int_0^{2\pi}\int_0^2 \sqrt{2}\,v^2\,dv\,du$$

$$= \int_0^{2\pi} \sqrt{2}\,\frac{v^3}{3}\Bigg|_{v=0}^{v=2} du = \int_0^{2\pi} \frac{8\sqrt{2}}{3}\,du = \frac{16\sqrt{2}}{3}\pi$$

So

$$\text{Ave} = \frac{1}{\text{Area}}\iint_S f\,dA = \frac{16\sqrt{2}\pi/3}{4\sqrt{2}\pi}$$

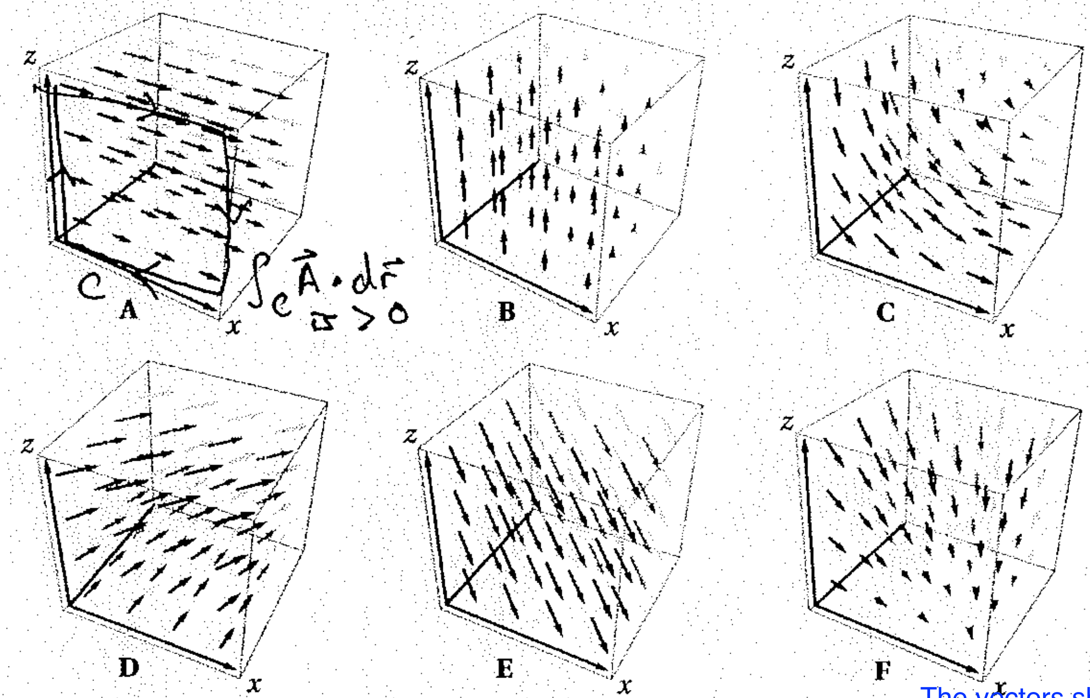
Average = $4/3$

13-a :

For fixed v we get circles about y-axis. For example, for $v=1$ we have $r(u,1) = (\cos u, 1, \sin u)$. This rules the 1^{st} and 4^{th} options.

For fixed u we get line segments starting from the origin. For example, for $u=0$ we have $r(0,v) = (v, v, 0)$. This rules out the 2^{nd} option, so the answer is the 3^{rd} option.

14. Here are plots of six vector fields on the box where $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. For each part, circle the best answer. **(1 point each)**



The vectors should be parallel to the xy-plane. The x component should increase as z value increases, and the y component should be 1 for each vector.

(a) The the vector field given by $\langle z, 1, 0 \rangle$ is: A B C D E F

(b) Exactly one of these vector fields has nonzero divergence. It is: A B C D E F

For this example, the divergence is generally: negative positive

The flow is sinking into the lower right corner.

(c) The vector field **A** is conservative: true false

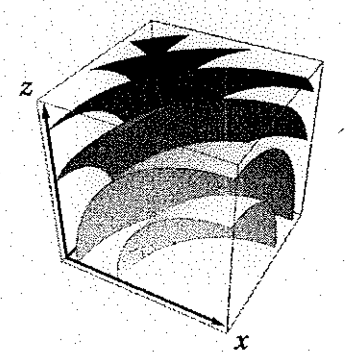
(d) Exactly one of the vector fields is constant, that is, independent of position. It is: A B C D E F

The vectors should have the exact same length and direction

(e) The vector field curl **C** is constant. The value of curl **C** is:

i	−**i**	**j**	−**j**	**k**	−**k**	**0**

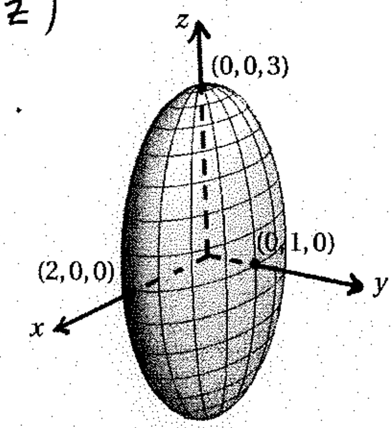
Right hand rule



(f) The vector field that is the gradient of a function f whose level sets are shown at right is: A B C D E F

The vectors should be perpendicular to the level sets.

15. Let S be the ellipsoid $\frac{x^2}{4}+y^2+\frac{z^2}{9}=1$ which is shown at right. Give a parameterization $\mathbf{r}\colon D \to \mathbb{R}^3$ for S,being sure to specify the domain D of the parameterization in the (u, v)-plane. **(? points)**



The transformation $(x, y, z) \longmapsto (2x, y, 3z)$ takes the unit sphere to this ellipsoid. Combining w/ our spherical coor, we can use

$D = \{\ 0 \leq u \leq \pi\ ,\ 0 \leq v \leq 2\pi\ \}$

$\mathbf{r}(u, v) = \langle 2\sin u \cos v\ ,\ \sin u \sin v\ ,\ 3\cos u \rangle$

16. For each transformation $\mathbb{R}^2 \to \mathbb{R}^2$ below, circle "yes" or "no" depending on whether or not it takes the rectangle $0 \leq u \leq 1$, $0 \leq v \leq 2$ in the (u, v)-plane to the parallelogram in the (x, y)-plane with vertices $(0,0)$, $(4,2)$, $(2,-2)$, and $(6,0)$. **(1 point each)**

a) yes **no** $T(u, v) = (2u+4v, -2u+2v)$

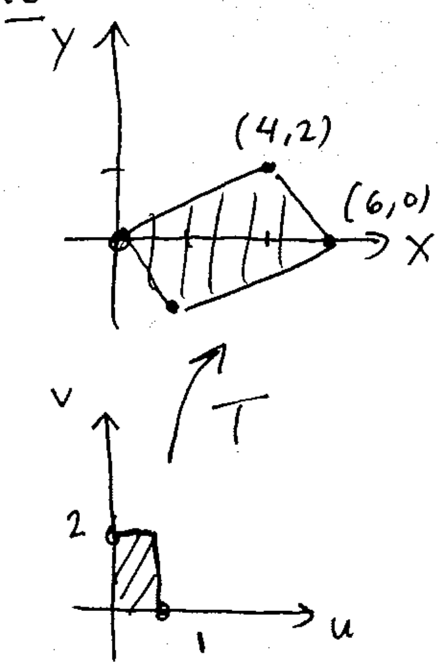
b) **yes** no $T(u, v) = (4u+v, 2u-v)$

c) yes **no** $T(u, v) = (4u+2v, 2u-2v)$

d) **yes** no $T(u, v) = (2u+2v, -2u+v)$

Scratch Space

16:



Any correct T must take (1,0) to either (2,-2) or (4,2) and take (0,2) to the other one.

a) $T(1,0) = (2,-2)$ $T(0,2) = (8,4)$

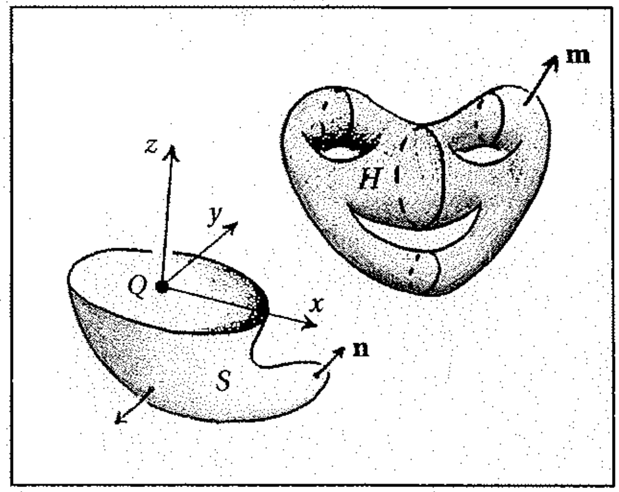
b) $T(1,0) = (4,2)$ $T(0,2) = (2,-2)$ ✓

c) $T(1,0) = (4,2)$ $T(0,2) = (4,-4)$

d) $T(1,0) = (2,-2)$ $T(0,2) = (4,2)$ ✓

17. Let $\mathbf{F} = \langle -x^3\cos^3(x) + y^3,\ y^3\sin^3(y) - x^3 \rangle$ and let C be the unit circle in the plane $\mathbb{R}^2$ oriented counter-clockwise. Determine whether the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is: negative zero positive **(2 points)**

18. Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. Suppose there is a positive charge Q placed at the origin and let $\mathbf{E}$ be the resulting electrical field. For each part, circle the correct answer. **(2 points each)**



(a) The flux $\iint_H \mathbf{E}\cdot\mathbf{m}\, dS$ is: negative zero positive

(b) The flux $\iint_S \mathbf{E}\cdot\mathbf{n}\, dS$ is: negative zero positive

(c) For $\mathbf{G} = \langle xy^2,\ yz,\ x+z \rangle$, the flux $\iint_H (\operatorname{curl}\mathbf{G})\cdot\mathbf{m}\, dS$ is:

negative zero positive

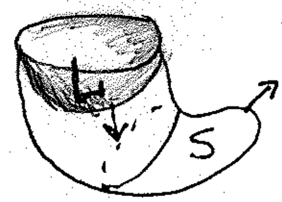
Scratch Space

17. By Mr. Green: $\int_C \vec{F}\cdot d\vec{r} = \iint_{Disc} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$

$= \iint_{Disc} -3x^2 - 3y^2\, dA < 0$ since the integrand is < 0 (except at the origin).

18 a. H does not contain the charge Q, so this is 0 by Gauss's Law.

18b. As $\operatorname{div}\vec{E} = 0$, flux through S is the same as that of the lower hemisphere with the same boundary. The field $\vec{E}$ is (outward) normal to L everywhere, and so flux is > 0.



18c. Since H is closed $\iint_H (\operatorname{curl}\vec{G})\cdot\vec{m}\, dS = 0$.