1. Let $A = (1,0,1)$, $B = (1,2,0)$, and $C = (2,3,1)$. For each part, circle the best answer. **(1 point each)**

(a) Let θ be the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. The value of θ is:

$\theta = 0$	$0 < \theta < \pi/2$	$\theta = \pi/2$	$\pi/2 < \theta < \pi$	$\theta = \pi$

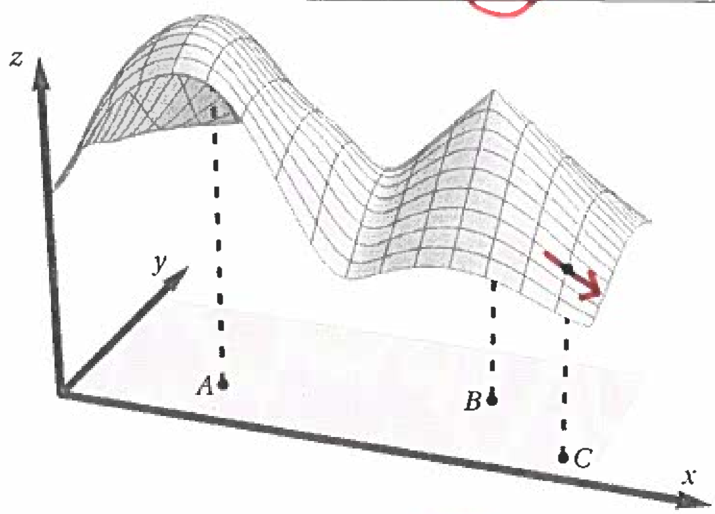
(b) The area of the triangle formed by these points is:

$\frac{\sqrt{14}}{2}$	$\sqrt{14}$	7	14

(c) Let ℓ be the line through B and C. The distance from A to ℓ is:

$\frac{\sqrt{14}}{2}$	$\frac{\sqrt{14}}{\sqrt{3}}$	$\frac{\sqrt{14}}{\sqrt{5}}$	$\frac{\sqrt{14}}{\sqrt{15}}$

2. Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ whose graph is shown at right. Let A and B be the points in $\mathbb{R}^2$ corresponding to the two "peaks" of the graph, and C be the point in $\mathbb{R}^2$ corresponding to the dot on the graph. For each part, circle the answer that is most consistent with the picture. **(1 point each)**



(a) At the point A, the function g is:

continuous	differentiable	both	neither

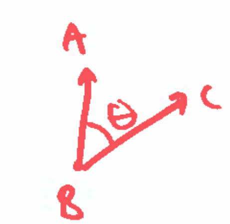
(b) At the point B, the function g is:

continuous	differentiable	both	neither

(c) At the point C, the function $\frac{\partial g}{\partial x}$ is:

negative	zero	positive

Scratch Space



$\cos\theta = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\|\vec{BC}\|} = \frac{0-2+1}{\sqrt{5}\sqrt{3}} = -1/\sqrt{15}$

$\vec{BA}\times\vec{BC} = \langle -2-1, -(0-1), 0--2\rangle = \langle -3, 1, 2\rangle$

$\| \quad \| = \sqrt{9+1+4} = \sqrt{14} \rightsquigarrow$ area $= \frac{1}{2}\sqrt{14}$

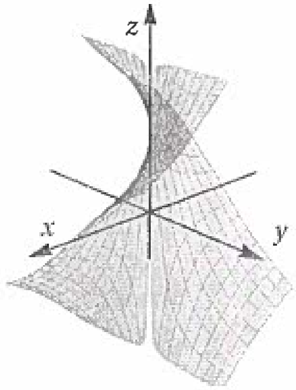
$\vec{BA} = \langle 0,-2,1\rangle$

$\vec{BC} = \langle 1,1,1\rangle$

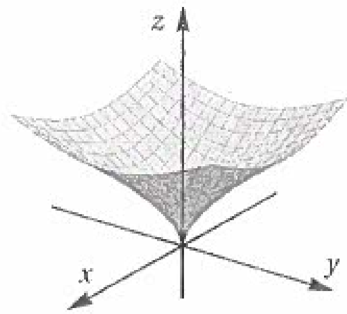
$d = |BA|\sin\theta = \frac{|\vec{BA}\times\vec{BC}|}{|\vec{BC}|} = \frac{\sqrt{14}}{\sqrt{3}}$

3. For each of the given functions, label the box below the picture corresponding to its graph. **(1 point each)**

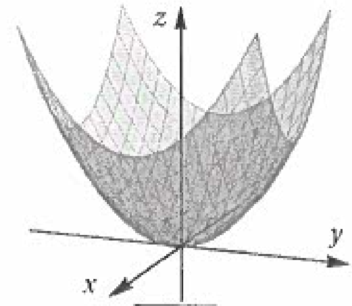
(A) $|x| + |y|$ (B) $x^2 + y^2$ (C) $\dfrac{xy^2}{x^2 + y^2}$



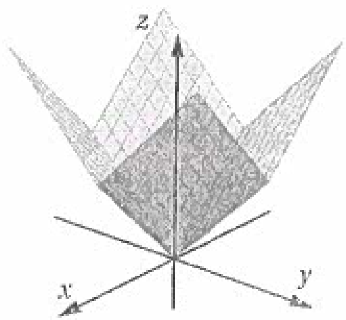
$\square$



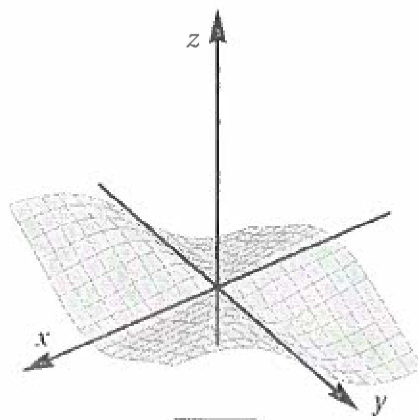
$\square$



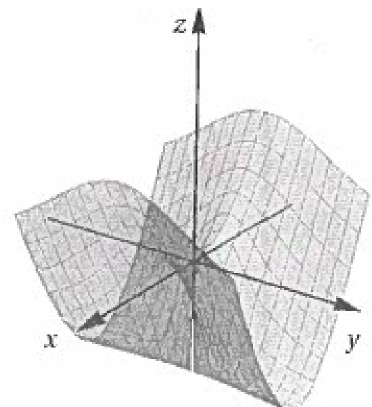
$\boxed{B}$



$\boxed{A}$



$\boxed{C}$



$\square$

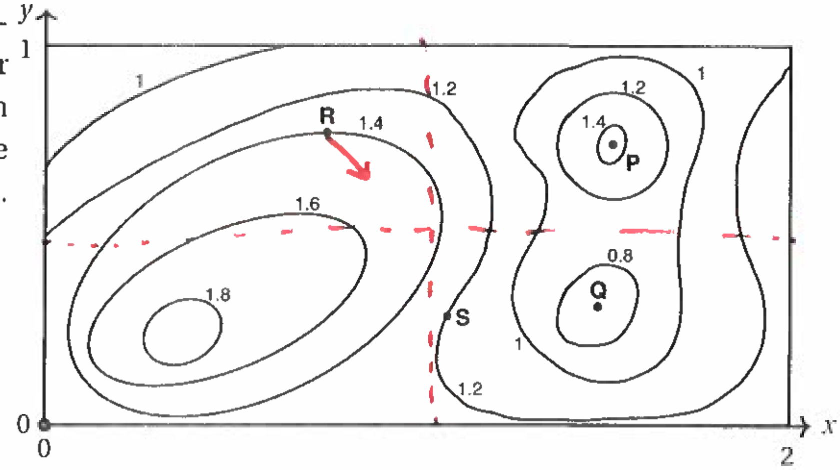
4. It is raining on a hill whose height is given by the function $h(x, y) = 20 - 2x^2 - 3y^2$. Assume that water always flows downhill, in the direction where the height of the hill decreases most quickly. At the point $(1, 1)$ what is the direction (in the xy-plane) in which the water will flow? **(2 points)**

At $(1, 1)$ the rain water flows in direction of $\boxed{\langle \quad 4 \quad , \quad 6 \quad \rangle}$

Scratch Space

$\nabla h = \langle -4x, -6y \rangle$

5. A rectangular garbage container with dimensions 2 meter high by 2 meter wide by 1 meter deep is partially filled with trash. The function $f(x, y)$ describes the height (in meters) of the trash; a contour map of f is shown to the right. For each part below, circle the best answer.



(a) Classify the behavior at the given points. **(1 point each)**

At P:	f has a local min	f has a local max	f has a saddle point	P is not a critical point
At Q:	f has a local min	f has a local max	f has a saddle point	Q is not a critical point
At R:	f has a local min	f has a local max	f has a saddle point	R is not a critical point

(b) $\nabla f(R) \approx$ $\langle 0, -3.5\rangle$ $\langle 0, -1.2\rangle$ $\langle 0, -0.2\rangle$ $\langle 0, 0\rangle$ $\langle 0, 0.2\rangle$ $\langle 0, 1.2\rangle$ $\langle 0, 3.5\rangle$ **(1 point)**

(c) Let $\mathbf{u}$ be a unit vector in the direction of $\overrightarrow{RS}$. The directional derivative $D_{\mathbf{u}} f(R)$ is:

negative zero positive **(2 points)**

(d) The volume of trash in the container (m^3) is $\approx$ 1.1 2.5 3.8 5.1 7.2 **(2 points)**

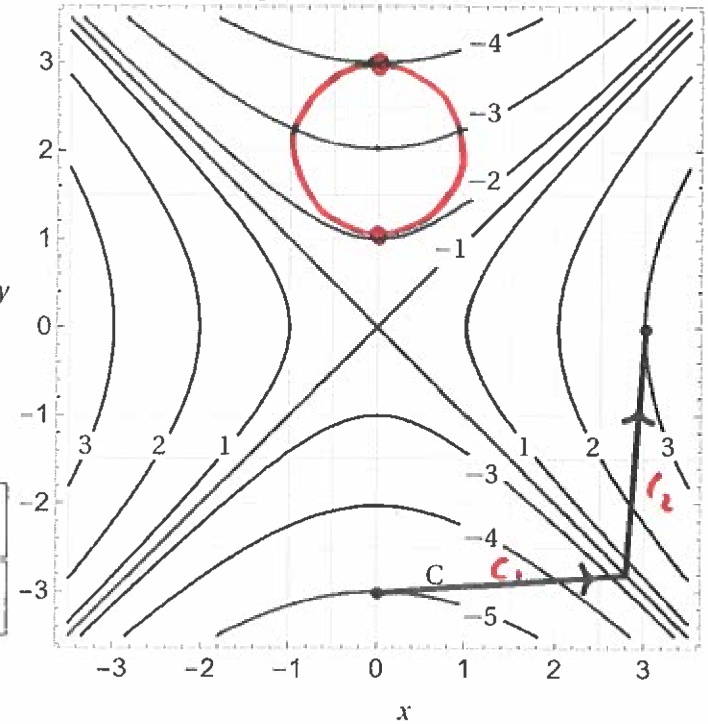
Scratch Space

$$f_y(R) \approx \frac{1}{2}\left(\frac{1.4-1.6}{1/4} + \frac{1.2-1.4}{1/8}\right) = \frac{1}{2}\left(4\left(\frac{-2}{10}\right) + 8\left(\frac{-2}{10}\right)\right) =$$

$$= \frac{1}{2}\left(-\frac{8}{10} - \frac{16}{10}\right) = \frac{1}{2}\left(-\frac{24}{10}\right) = -1.2$$

$$\text{Vol} \approx \frac{1}{2}\left[1.7 + 1.4 + 1.4 + 1\right] = \frac{1}{2}\left[2.7 + 2.8\right] = \frac{1}{2}\left[5.5\right] = 2.75$$

6. The contour plot of a differentiable function f is shown below. For each part, circle the best answer.



(a) Estimate $\int_C f\,ds$: **(2 points)**

−9	−5.5	−0.6	0	0.6	5.5	9

(b) Estimate $\int_C \nabla f \cdot d\mathbf{r}$:

−16	−8	−4	0	4	8	16

(2 points)

(c) Find the points on the curve $x^2+(y-2)^2=1$ where f has max/min values. **(2 points)**

Max value = $\boxed{2}$ at the point(s) $\boxed{(0,1)}$

Min value = $\boxed{-4}$ at the point(s) $\boxed{(0,3)}$

(d) What is the absolute maximum value of f on the region $D=\{x^2+y^2<1\}$? Write *DNE* if none exists.

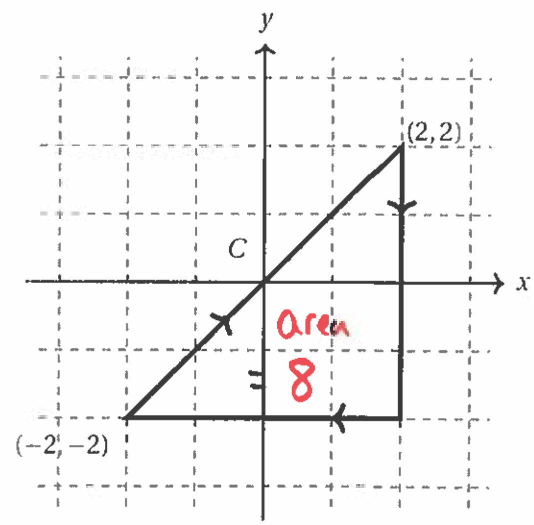
Max value on D = $\boxed{\text{DNE}}$ **(1 point)**

$P_y = -1 \qquad Q_x = 2$

7. Let $\mathbf{F} = \left\langle 3x^2y - y,\ x^3 + 2x + \sqrt{1+y^4} \right\rangle$. For C, the curve shown at right, compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$:

−24	−16	~~−8~~	0	8	16	24

(2 points)



Scratch Space

$\int_{C_1} f\,ds = \text{length}(C_1)\cdot\text{avg}(f) \approx (3)(-4)$

$\int_{C_2} f\,ds = \text{length}(C_2)\,\text{avg}(f) \approx (3)(2)$

$\int f \approx -12+6 = -6$

$\int_C \nabla f\, d\mathbf{r} = 3-(-5) = 8$

8. Determine the limits in the problems below. Be sure to *explain your reasoning* for full credit. If a limit does not exist, write "DNE" in the box provided. **(2 points each)**

(a) Determine $\lim_{(x,y)\to(0,0)} \frac{xy+x^2}{x^2+y^2}$.

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{xy+x^2}{x^2+y^2} = \text{DNE}}$$

along $x=0$: $\lim_{y\to 0} \frac{0+0}{0+y^2} = \lim_{y\to 0} 0$

along $x=y$ $\lim_{x\to 0} \frac{x^2+x^2}{x^2+x^2} = \lim_{x\to 0} 1 = 1$

diff. vals $\Rightarrow$ DNE

(b) Determine $\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2-x^3y}$.

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2-x^3y} = \frac{1}{2}}$$

$= \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2(2-xy)} = \lim_{(x,y)\to(0,0)} \frac{1}{2-xy}$

$= \frac{1}{2}$

Scratch Space

9. The function $f(x, y)$ describes the temperature (°C) in a region R in the plane, so that $f(x, y)$ is the measured temperature at position (x, y). Some measured values of f and its rates of change are given in the following table. Assuming that f is differentiable, use this data to approximate the temperature at $(1.5, 3.1)$. **(2 points)**

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$
(1,3)	4	2	3
(0.5,0.1)	−5	−1	−6

$f(1+\Delta x, 3+\Delta y) \approx f(1,3) + f_x(1,3)\cdot(\Delta x) + f_y(1,3)\Delta y$

$= 4 + 2(0.5) + 3(0.1) = 5.3$

$\Delta x = 0.5$

$\Delta y = 0.1$

Temperature at (1.5, 3.1) is ≈ $\boxed{5.3\ °C}$

10. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function of x and y where $x(s,t) = t\cos(2s) - e^{2t}$ and $y(s,t) = t\sin(2s) + e^{t\sin(t)}$.
Let $g(s,t) = f(x(s,t), y(s,t))$. Use the table of values on the right, to calculate $g_s\left(\frac{\pi}{4}, \pi\right)$. (5 points)

	g	f	f_x	f_y
$(\pi/4, \pi)$	2	−1	3	5
$(-e^{2\pi}, \pi+1)$	4	2	−3	−2

Chain rule:

$g_s = f_x\, x_s + f_y\, y_s$

$x(\pi/4, \pi) = \pi\cos(\pi/2) - e^{2\pi}$
$= -e^{2\pi}$

$y(\pi/4, \pi) = \pi\sin(\pi/2) + e^{\pi \sin \pi}$
$= \pi + 1$

$x_s = -2t\sin(2s)$

$y_s = 2t\cos(2s)$

$x_s(\pi/4, \pi) = -2\pi\sin(\pi/2)$
$= -2\pi$

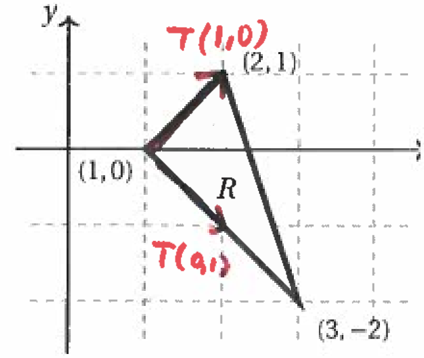
$y_s(\pi/4, \pi) = 2\pi\cos(\pi/2) = 0$

$g_s(\pi/4, \pi) = -3\cdot(-2\pi) + (-2)(0)$

$g_s\left(\frac{\pi}{4}, \pi\right) = \boxed{6\pi}$

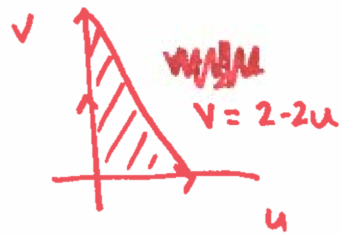
Scratch Space

11. Using the transformation $T(u,v) = \langle u+v+1, u-v \rangle$, rewrite the integral $\iint_R x\,dA$ as an iterated integral over a subset S in the uv-plane with $T(S) = R$. *Do not evalute the integral.* **(5 points)**



$T(1,0) = (2,1)$

$T(0,1) = (2,-1)$



$|J| = \left| \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \right| = 2$

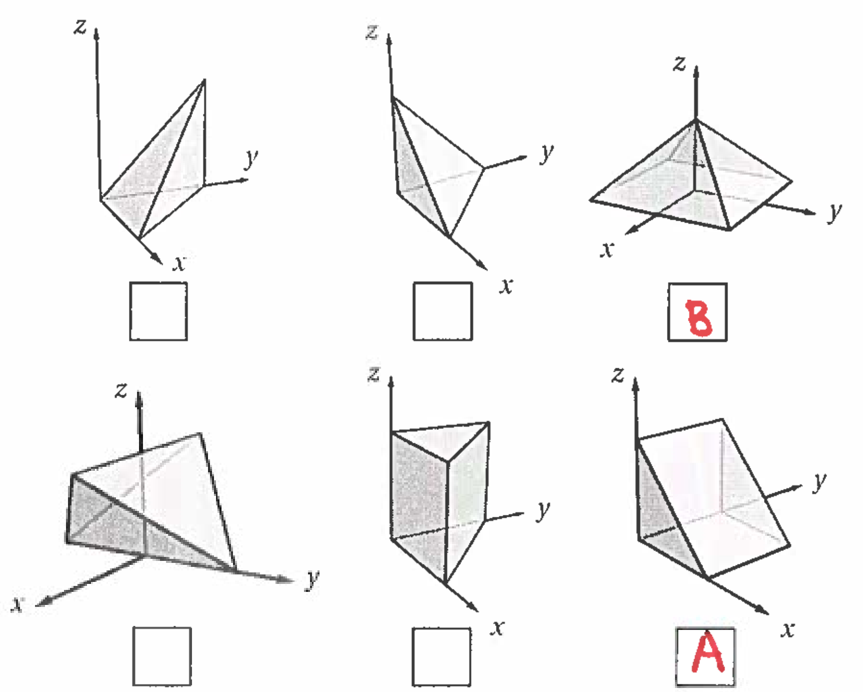
$$\iint_R x\,dA = \boxed{\int_0^2 \int_0^{1-v/2} (u+v+1)\; 2 \quad du\,dv}$$

Note: The order of integration is already determined.

12. For each of the given integrals, label the box below the picture of the corresponding region of integration.
(2 points each)

(*A*) $\displaystyle\int_0^1 \int_0^1 \int_0^{1-z} f(x,y,z)\,dx\,dy\,dz$

(*B*) $\displaystyle\int_0^1 \int_{-1+z}^{1-z} \int_{-1+z}^{1-z} g(x,y,z)\,dx\,dy\,dz$

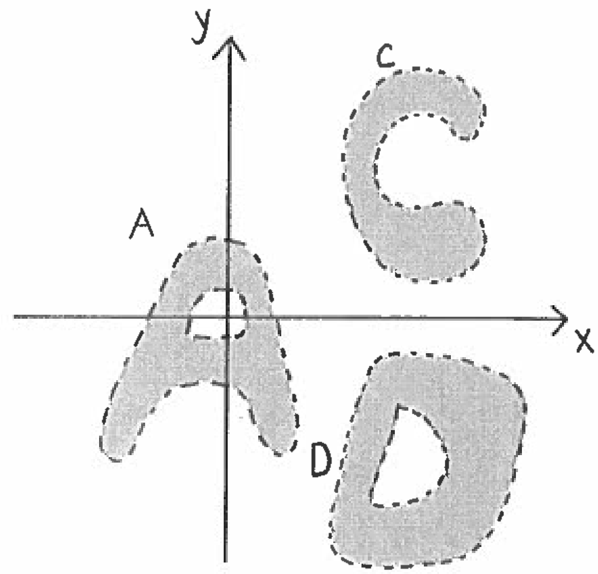


13. Consider a 2D vector field **F**, whose domain consists of those points of $\mathbb{R}^2$ which are not the origin.

Let $\mathbf{F} = \langle P, Q \rangle$.

Assume that $P_y = Q_x$ on the domain of definition of **F**, and assume further that **F** is <u>not conservative</u> on its domain.

Given are the (interiors of the) three regions A, C, D sketched in the picture at right. From this information, which of the following statements are true? Mark all the true statements. **(3 points)**



- [ ] For every closed continuous curve γ contained in A, the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ must be zero.
- [x] For every closed continuous curve γ contained in C, the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ must be zero.
- [ ] There exists some closed continuous curve γ contained in C, such that the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ must be non-zero.
- [x] For every closed continuous curve γ contained in D, the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ must be zero.
- [ ] There exists some closed continuous curve γ contained in D, such that the line integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ must be non-zero.
- [ ] No such vector field exists.

Scratch Space

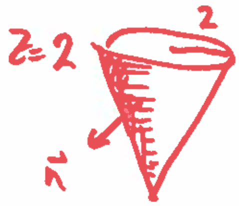
14. Let $\mathbf{F} = (1 + x + yz)\mathbf{i} + 2y\mathbf{j} + (z + yx)\mathbf{k}$.

(a) Compute curl(**F**). **(1 point)**

$$\begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 1+x+yz & 2y & z+yx \end{vmatrix} \qquad \langle x-0, -(y-y), 0-z \rangle$$

$\boxed{\text{curl}(\mathbf{F}) = \langle \; x \;, \; 0 \;, \; -z \; \rangle}$.

(b) Let S be the portion of the cone $z = \sqrt{x^2 + y^2}$ with $z \leq 2$, with outward pointing normal vector. Compute the flux of curl(**F**) through S. **(5 points)**



D= disk

radius 2

~~upward~~ normal

downward

Stokes $\Rightarrow \iint_S \text{curl}\vec{F}\cdot\vec{n}\, dA$

$= \iint_D \text{curl}\vec{F}\cdot\vec{n}\, dA$

$= \iint_D \langle x, 0, -z\rangle \langle 0, 0, -1\rangle dA$

$= \iint_D z\, dA = \iint_D 2\, dA = 2(\pi(2)^2)$

$\boxed{\text{flux} = 8\pi}$

(c) Let E be the sphere $x^2 + y^2 + z^2 = 1$ with outward pointing normal. Compute the flux of **F** through E. **(3 points)**

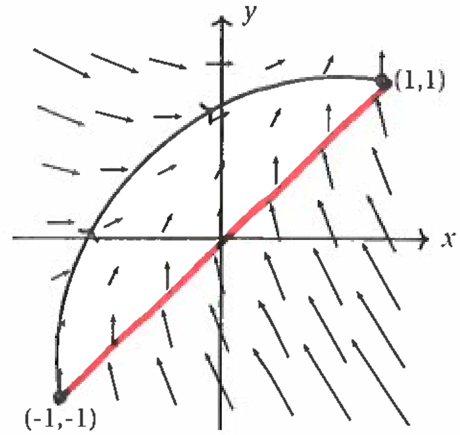
divergence theorem: B = unit ball

$$\iint_E \vec{F}\cdot\vec{n}\, dA = \iiint_B \text{div}\vec{F}\, dV = \iiint_B (1+2+1)\, dV = 4\cdot\left(\frac{4}{3}\pi\right)$$

$\boxed{\text{flux} = 16/3\ \pi}$

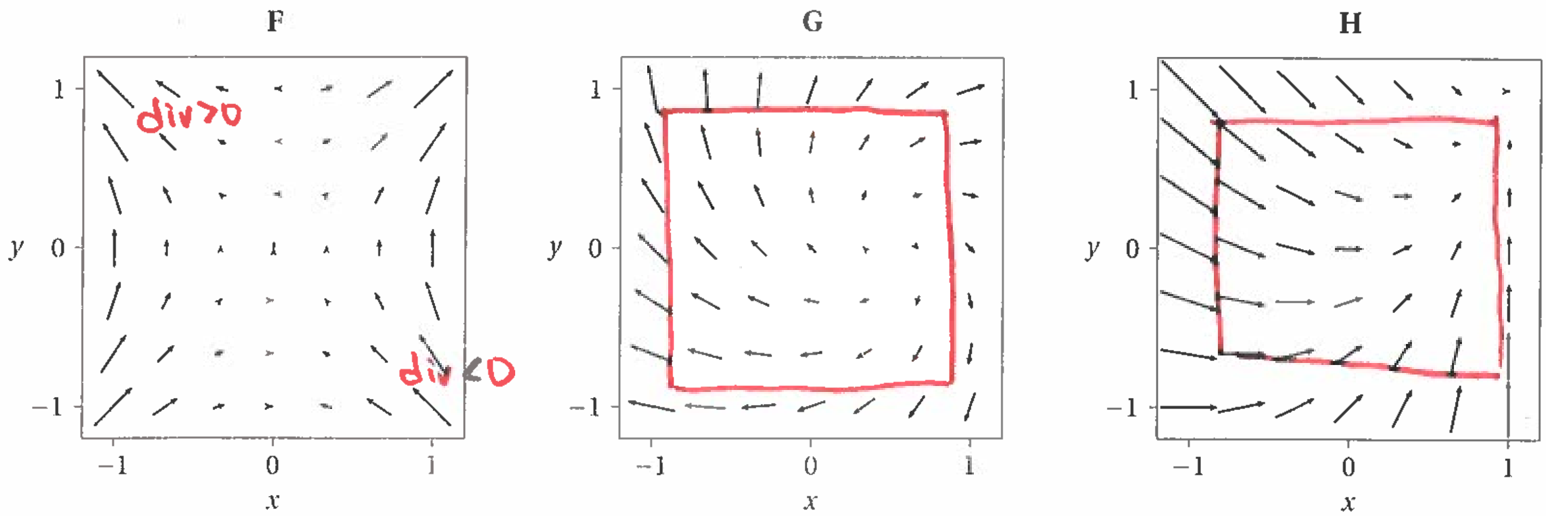
(d) Is **F** conservative? | Yes | No | **(1 point)**

15. A conservative force field **F** is shown at right. Compute the work done by **F** to move a particle from $(-1,-1)$ to $(1,1)$ along the indicated path. For scale, $\mathbf{F}(0,0) = \langle 0, 0.2 \rangle$.



-1	$-\frac{2}{5}$	$-\frac{2\sqrt{2}}{5}$	0	$\frac{2\sqrt{2}}{5}$	$\frac{2}{5}$	1	**(2 points)**

16. Three vector fields are shown below, **exactly one** of which is conservative. For each of the following questions, circle the best answer.



(a) The conservative field is: F G H **(2 points)**

(b) The vector field $(1-x)\mathbf{i} + (x-y)\mathbf{j}$ is: F G H **(1 point)**

(c) The vector field curl(**G**) is constant. The value of curl(**G**) at any point is:

$\langle 0,0,-1\rangle$ $\langle 0,0,0\rangle$ $\langle 0,0,1\rangle$ **(1 point)**

(d) **Exactly one** of these vector fields has nonconstant divergence. Circle it: F G H **(1 point)**