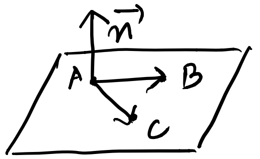
1. Consider the points $A = (0,0,2)$, $B = (1,0,3)$, and $C = (0,1,3)$ in $\mathbb{R}^3$.

(a) Compute the vectors $\mathbf{v} = \overrightarrow{AB}$ and $\mathbf{w} = \overrightarrow{AC}$. **(2 points)**

$$v = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (1,0,3) - (0,0,2) = (1,0,1)$$

$$w = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = (0,1,3) - (0,0,2) = (0,1,1)$$

(b) Find a normal vector $\mathbf{n}$ to the plane P containing the points A, B, C. **(3 points)**

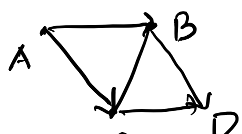


$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \vec{j} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{k}$$

$$= (-1,-1,1)$$

(c) Find the area of the triangle spanned by A, B, C. **(2 points)**

$$|\vec{v} \times \vec{w}| = |\overrightarrow{AB} \times \overrightarrow{AC}| = \text{Area of the } \parallel gm \ ABCD$$



& Area $\triangle ABC = \frac{1}{2}$ Area $\parallel gm$ ABCD

So, Area $\triangle ABC = \frac{1}{2} |(-1,-1,1)|$

$$= \frac{1}{2}\sqrt{(-1)^2 + (-1)^2 + (1)^2} = \frac{1}{2}\sqrt{3}$$

(d) Find an equation which describes P. If you can't do (b), take $\mathbf{n} = (1,-2,-1)$. **(1 point)**

To describe the eqn. of the plane P we need
• a normal vector to the plane • a point on the plane

$\vec{n} = (-1,-1,1)$ $\qquad$ $A = (0,0,2)$.

Eqn. $\longrightarrow \vec{n} \cdot (x-0, y-0, z-2) = 0$

$\Rightarrow -1 \cdot x + -1 \cdot y + 1 \cdot (z-2) = 0 \Rightarrow z - x - y = 2$

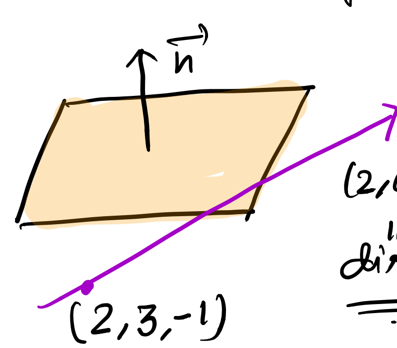
(e) Consider the line L given by the parameterization $\mathbf{r}(t) = (2+2t, 3, -1+2t)$. Is L parallel to the plane P? Why or why not? **(2 points)**

$r(t) = (2+2t, 3, -1+2t) = (2,3,-1) + t(2,0,2)$

• this implies that the direction vector of the line is $(2,0,2)$.

So, the line is parallel to the plane if and only if

$(2,0,2) \cdot \vec{n} = 0$



(2,0,2) = direction

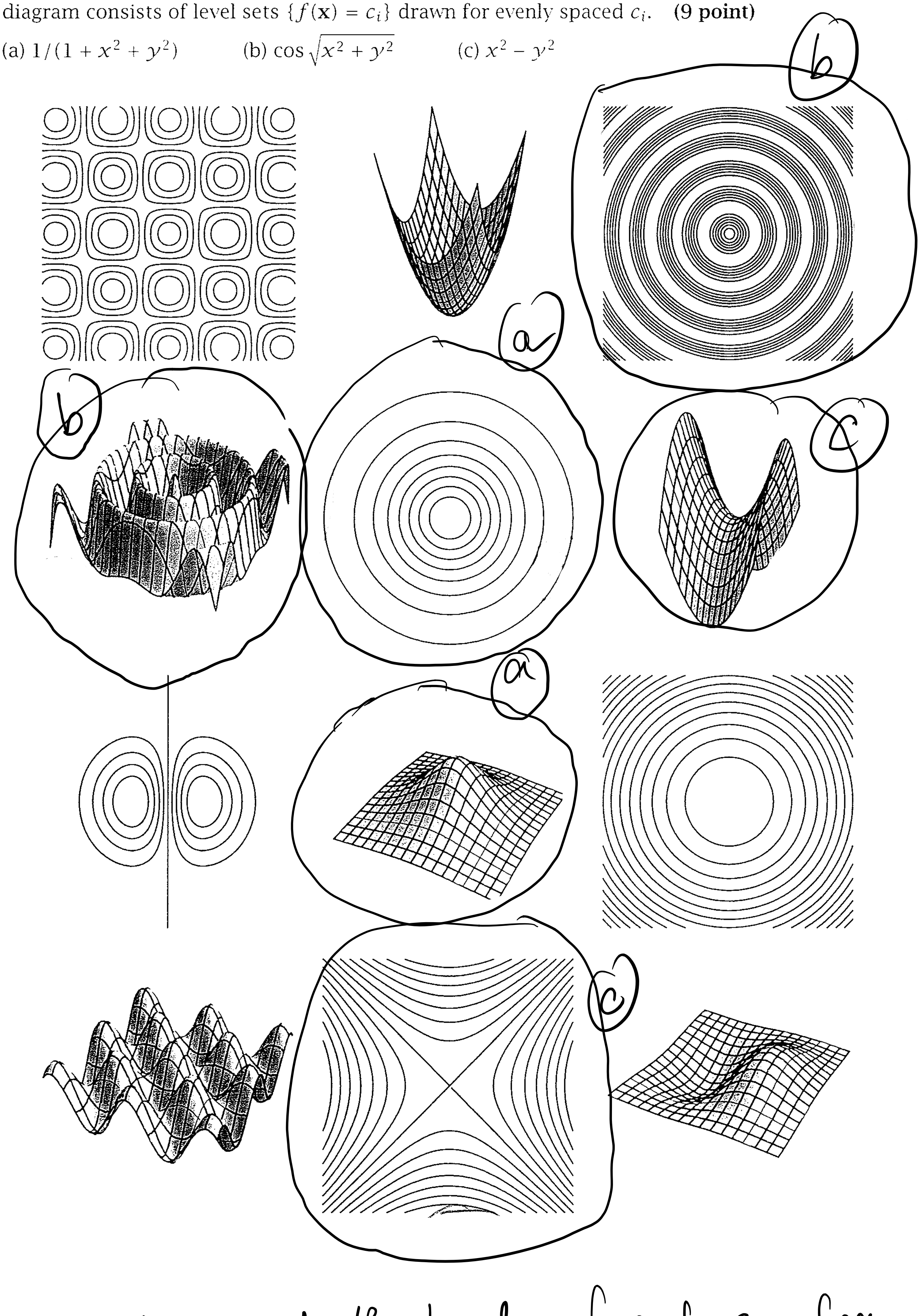
And indeed $(2,0,2) \cdot \vec{n}$

$= -1 \cdot 2 + -1 \cdot 0 + 1 \cdot 2 = 0$

So the line is parallel to

2. Match the following functions with their graphs and level set diagrams. Here each level set diagram consists of level sets $\{f(\mathbf{x}) = c_i\}$ drawn for evenly spaced c_i. **(9 point)**

(a) $1/(1+x^2+y^2)$ (b) $\cos\sqrt{x^2+y^2}$ (c) x^2-y^2



look at the last few pages for explanation.

3. Consider the function $f(x,y) = \frac{y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Compute the following limit, if it exists. **(5 points)**

$$\lim_{(x,y)\to(0,0)} f(x,y)$$ to $(0,0)$

Idea: We will calculate this limit along two directions, and show that they are not equal. Hence, the limit can't exist.

- $\lim_{t\to 0} f((t,0)) = \lim_{t\to 0} \frac{0}{t^2+0} = \lim_{t\to 0} 0 = 0.$
- $\lim_{t\to 0} f((0,t)) = \lim_{t\to 0} \frac{t^2}{0+t^2} = \lim_{t\to 0} \frac{t^2}{t^2} = \lim_{t\to 0} 1 = 1$

Now observe that $\lim_{t\to 0} (t,0) = (0,0) = \lim_{t\to 0} (0,t)$

and $1 \neq 0$ so $\lim_{(x,y)\to(0,0)} f(x,y)$ can't exist.

4. Consider the composition of the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $x, y\colon \mathbb{R}^2 \to \mathbb{R}$, that is

$$h(s,t) = f\big(x(s,t), y(s,t)\big)$$

Compute $\frac{\partial h}{\partial s}(1,2)$ using the chain rule and the table of values below. **(5 points)**

input	x	y	f	$\frac{\partial x}{\partial s}$	$\frac{\partial y}{\partial s}$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
(0,1)	1	1	4	1	2	7	3
(1,1)	1	2	6	1	1	6	2
(1,2)	0	1	5	2	3	5	1
(2,3)	2	3	4	0	1	4	1

By Chain rule,

for any $(a,b) \in \mathbb{R}^2$.

$$\frac{\partial h}{\partial s}(a,b) = \frac{\partial f}{\partial x}\big(x(a,b), y(a,b)\big)\cdot\frac{\partial x}{\partial s}(a,b) + \frac{\partial f}{\partial y}\big(x(a,b), y(a,b)\big)\cdot\frac{\partial y}{\partial s}(a,b). \quad (*)$$

The problem is asking for $\frac{\partial h}{\partial s}(1,2)$ so, we need to evaluate $(*)$ at $(a,b) = (1,2)$.

So,

$$\frac{\partial h}{\partial s}(1,2) = \frac{\partial f}{\partial x}\big(x(1,2), y(1,2)\big)\frac{\partial x}{\partial s}(1,2) + \frac{\partial f}{\partial y}\big(x(1,2), y(1,2)\big)\frac{\partial y}{\partial s}(1,2)$$

$$= \frac{\partial f}{\partial x}(0,1)\cdot 2 + \frac{\partial f}{\partial y}(0,1)\cdot 3 = 7\cdot 2 + 3\cdot 3 = 14+9$$

$$= 23.$$

5. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + \dfrac{x}{y}$.

(a) Compute the partial derivatives f_x, f_y and f_{xy}. **(3 points)**

$$f_x = \frac{\partial f}{\partial x} = 2x + \frac{1}{y} \qquad f_y = \frac{\partial f}{\partial y} = 0 + \frac{x}{y^2}(-1). = \frac{-x}{y^2}.$$

$$f_{xy} = \frac{\partial}{\partial y}\left(2x + \frac{1}{y}\right) = 0 - \frac{1}{y^2} = -\frac{1}{y^2}$$

(b) Is f differentiable at $(2,1)$? Why or why not? **(2 points)**

Yes, because both partial derivatives of f (f_x & f_y) exist, and are continuous near $(1,2)$

$$\left(f_x = 2x + \frac{1}{y},\ f_y = -\frac{x}{y^2}\right)$$

(c) Give the linear approximation of f at the point $(2,1)$: **(2 points)**

$$f(2+\Delta x, 1+\Delta y) \approx f(2,1) + \frac{\partial f}{\partial x}(2,1)\,\Delta x + \frac{\partial f}{\partial y}(2,1)\,\Delta y$$

$$= 6 + 5\Delta x + (-2)\Delta y$$

$$= 6 + 5\Delta x - 2\Delta y.$$

(d) Give the equation of the tangent plane to the graph of f at $(2,1,6)$. **(2 points)**

Best Linear approximation of f at $(2,1)$ is given by

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$= 6 + 5(x-2) - 2(y-1)$$

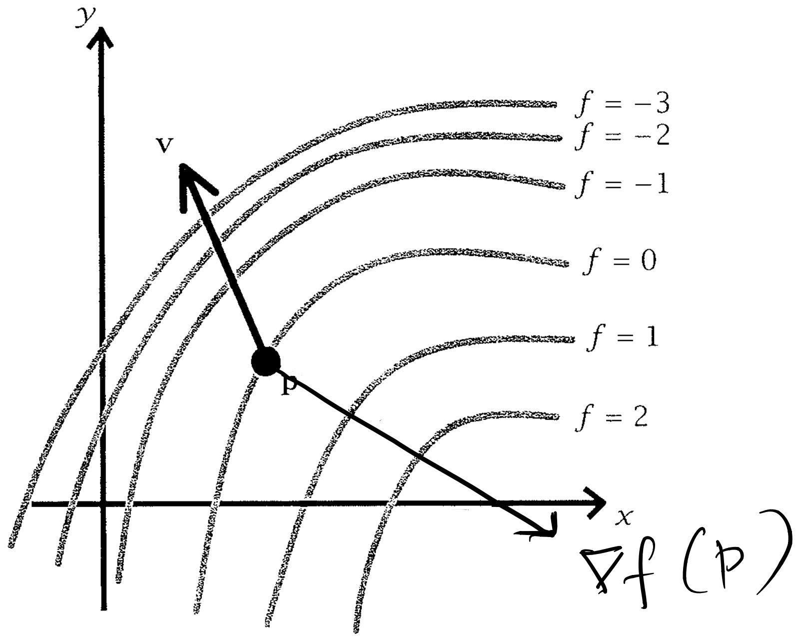
So the tangent plane is given by

$$z = L(x,y) = 6 + 5(x-2) - 2(y-1)$$

$$= 6 + 5x - 10 - 2y + 2 = -10 + 8 + 5x - 2y$$

$$= -2 + 5x - 2y$$

6. The picture below shows some level sets of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$.



(a) At the point **p** shown, determine the sign of each of the below quantities. **(1 points each)**

$f(\mathbf{p})$: positive negative 0

$f_x(\mathbf{p})$: positive negative 0

$f_y(\mathbf{p})$: positive negative 0

$f_{xx}(\mathbf{p})$: positive negative 0

$D_{\mathbf{v}}f(\mathbf{p})$: positive negative 0

(b) Draw $\nabla f(\mathbf{p})$ on the picture **(1 points)**.

Look at the last few pages for explanation.

Extra credit problem: Let $E\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $E(x,y) = 3x^2 + xy$. Find a $\delta > 0$ so that $|E(\mathbf{h})| < 0.01$ for all $\mathbf{h} = (x,y)$ with $|\mathbf{h}| < \delta$. Carefully justify why the δ you provide is good enough. **(3 points)**

Take $\delta = \frac{1}{100}$. $|\vec{h}| = \sqrt{x^2+y^2}$. Clearly, $\sqrt{x^2} \leq \sqrt{x^2+y^2}$ & $\sqrt{y^2} \leq \sqrt{x^2+y^2}$

So, $x < \sqrt{x^2+y^2} = |\vec{h}|$ & $y < |\vec{h}|$

So, if $|\vec{h}| < \delta$, then $|x| < \delta$ & $|y| < \delta$ aswell.

Then $|E(\vec{h})| = |3x^2 + xy| < |3x^2| + |xy| = 3|x|^2 + |x||y|$

Triangle inequality

and $3|x|^2 + |x||y| < 3\delta^2 + \delta\cdot\delta = 4\delta^2 = \frac{4}{10^4} < \frac{1}{100}$

So, this δ serves the purpose!

Explanation for problem 2:

(a) $\frac{1}{1+x^2+y^2} = f(x,y)$

Observations about the function.

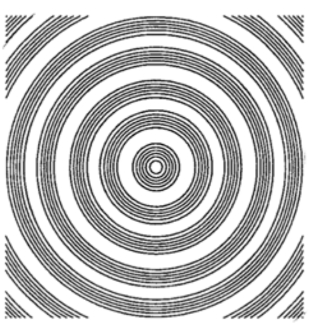
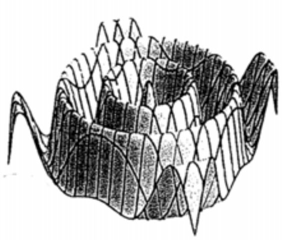
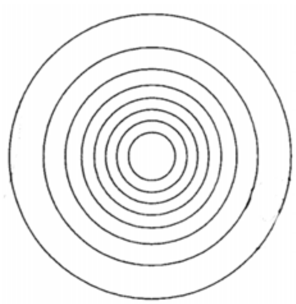
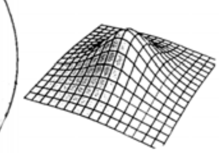
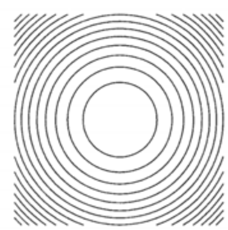
1a The function is symmetric around the origin.
i.e if $|(x_1,y_1)| = |(x_2,y_2)|$ then $f(x_1,y_1) = f(x_2,y_2)$.

2a The function decreases as $|x|$ & $|y|$ increases.

3a As $|x|$ & $|y|$ become very large i.e $|x|,|y| \to \infty$ the function approaches 0 and behaves almost like the constant function 0 (i.e rate of decrease is very less)

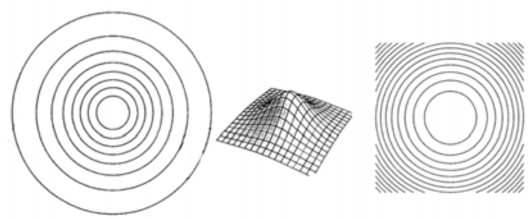
4a The rate of change of the function slows down as one moves away from the origin. i.e. the space between contours increase as we move away from origin.

1a eliminates everything except the following

Now we can use 2a to see what we can eliminate:

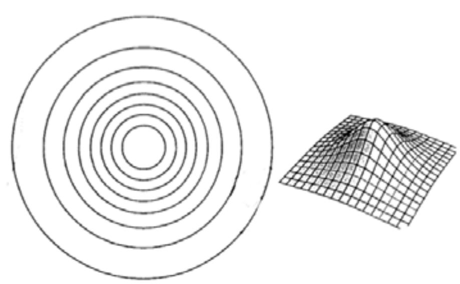
2a. eliminates the first two among the above 5 graphs. so, we are left with these three.



3a. does not elimate any from these 3 graphs.

4a. elimates the last one. (since the space between the contours decrease in this one).

So, we are left with the first two:

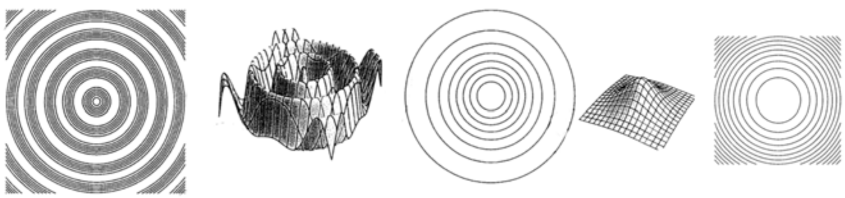


Its easy to see that they are indeed the level set and the graph of the given function.

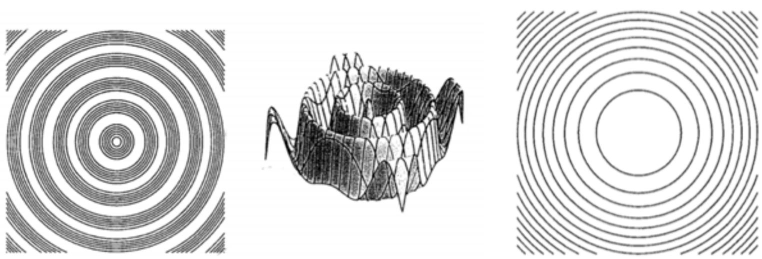
(moreover, after elimination you are left with these two options so they better be the correct options :)

(b) $\cos\sqrt{x^2+y^2} = h(x,y)$.

1b. The function is symmetric around the origin. i.e if $|(x_1,y_1)| = |(x_2,y_2)|$ then $h(x_1,y_1) = h(x_2,y_2)$.
Using this you are left with the following 5 pictures.



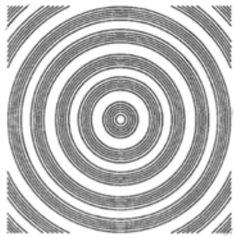
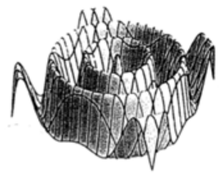
Now, the 3rd & the 4th are taken the function in (a), which means you are lef with the following three.



Among the above three there is just 1 graph. which is the middle picture so that better bee the graph of $h(x,y)$ and upon some thought its clear that its indeed the graph of $h(x,y)$.

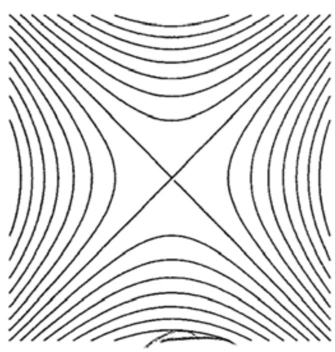
2b $h(x,y)$ is also periodic (w.r.t. $r=\sqrt{x^2+y^2}$) and the last picture is clearly not periodic so that can't bee the contour of the given function.

so, the correct pictures are

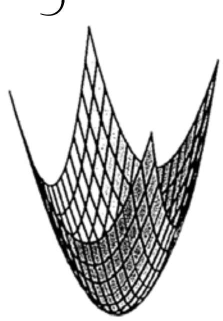
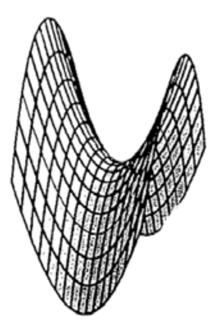
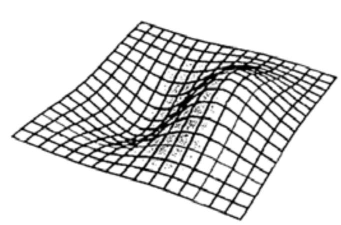
 

(c) $x^2 - y^2 = g(x,y)$.

1c note that if $x = \pm y$. then $g(x,y) = 0$.
This implies that the lines $x = y$ & $x = -y$ must lie in the contour map of $g(x,y)$.
And the only contour which has that is this one.



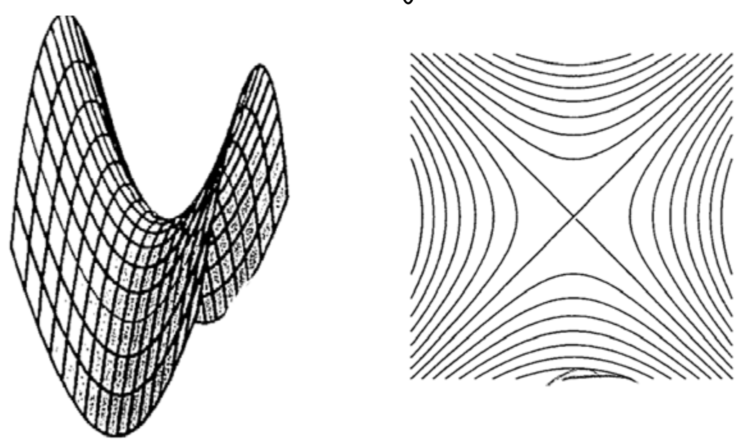
Among the graphs you are left with the following three graphs.

Now observe that the function $g(x,y)$ does not increase radially. i.e if $\sqrt{x_1^2+y_1^2} > \sqrt{x_2^2+y_2^2}$ then $f(x_1,y_1)$ might not be greater than $f(x_2,y_2)$, so this eliminates the first graph.

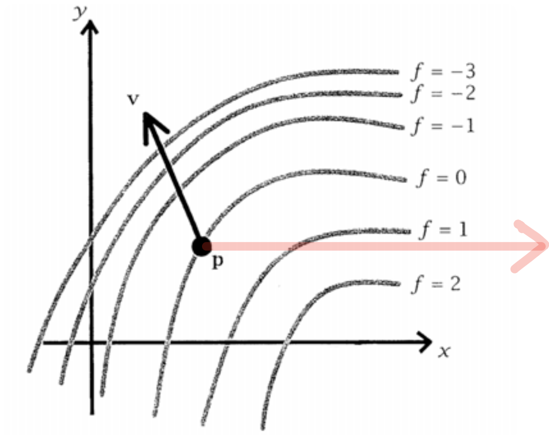
Now the fact that $g(x,y)$ is not bounded eliminates the third one.

So, the correct graph & level set are:

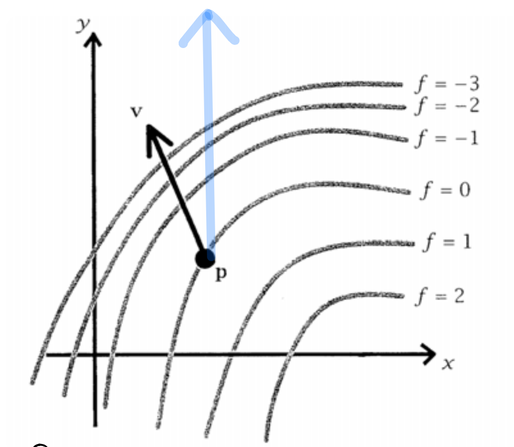


6. Explanation for problem 6.

(a) p lies on the level set $f=0$ so $f(p)=0$.



To evaluate $f_x(p)$ you need to see how the value of the function changes as one moves along the red line starting at p. From the level sets its clear that the function increases, so $f_x(p)>0$.



Similarly for $f_y(p)$ you need to ~~see~~ find the change along the blue line. The function decreases so $f_y(p)<0$.

for $D_{\vec{v}}f(p)$ you need to find the cange along the direction of $\vec{v}$. In this case too the function decreases so $D_v f(p) < 0$.

- The space between the level set increase as you go along the x-direction starting from point p (i.e along the red line), which is to say that it takes longer to have the same change in the functions value as one goes in that direction, so the rate of increase is actually slowing down. This implies that $f_{xx}(p)$ which measures the rate of change of the first partial derivative (f_x) is negative, i.e. the rate of increase is slowing down.

(b) The direction of $\nabla f(p)$ is the direction in which f increases most rapidly starting at p. i.e the direction in which the contours have very less space between them plus the function also increases in that direction. This direction in the case of the given function is the direction maked with a pink arrow in the diagram below.