MIDTERM 1 2011
SOLUTIONS

(1) *Solution.*

(a) $\mathbf{v} = \langle 2,2,3\rangle - \langle 0,1,2\rangle = \langle 2,1,1\rangle$ and $\mathbf{w} = \langle -1,3,4\rangle - \langle 0,1,2\rangle = \langle -1,2,2\rangle$.

(b) We may take the cross product of any two vectors on the plane. For example, we may take the cross product of $\mathbf{a}$ and $\mathbf{b}$ to get

$$\mathbf{n} \;=\; \mathbf{a}\times\mathbf{b} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 3 & 6 & 1 \end{vmatrix} \;=\; \begin{vmatrix} 2 & 0 \\ 6 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}\mathbf{k} \;=\; \langle 2,-1,0\rangle.$$

(c) The vector equation for the plane is $\mathbf{u}\cdot\big(\langle x,y,z\rangle - P\big) = 0$. Writing this all out, we get

$$\langle 3,2,4\rangle \cdot \big\langle x-2, y-(-1), z-1\big\rangle \;=\; 3(x-2)+2(y+1)+4(z-1) \;=\; 0.$$

Distributing and simplifying gives the equation $3x+2y+4z=8$.

(d) The absolute value (magnitude) of the cross product of $\mathbf{v}$ and $\mathbf{w}$ measures the area of the parallelogram with sides $\mathbf{v}$ and $\mathbf{w}$. □

(2) *Solution.*

(a) To find the midpoint M, we can "average" P and Q to get

$$M \;=\; \frac{1}{2}(P+Q) \;=\; \Big(\frac{1+3}{2}, \frac{3-3}{2}, \frac{-2+4}{2}\Big) \;=\; (2,0,1).$$

We can also add $\frac{1}{2}$ of the vector $\vec{PQ} = Q - P = \rangle 2,-6,6\rangle$ to P to get

$$M \;=\; (1,3,2) + \frac{1}{2}(2,-6,6) \;=\; (2,0,1).$$

(b) We have $\mathbf{r}(t) = \langle 1,0,2\rangle + t\langle 2,-1,1\rangle = \langle 1+2t, -t, 2+t\rangle$, so each point x,y,z on L looks like $x = 1+2t$, $y=-t$, $z = 2+t$ for some t. Substitute these for x y and z into the equation of the plane to get an equation

$$3(1+2t) - 2(-t) + (2+t) \;=\; 14.$$

Solving this equation to t, we get $t=1$. Plugging this value of t back into $\mathbf{r}(t)$ gives us the point $\mathbf{r}(1) = (3,-1,3)$, which is on both the line L and the plane. □

(3) *Solution.* Let's try taking the limit along two paths through the origin. We first approach along the y-axis: set $x=0$ to get $f(0,y) = \frac{0}{y^2} = 0$ and take the limit as y goes to zero to get

$$\lim_{y\to 0} f(0,y) \;=\; \lim_{y\to 0} 0 \;=\; 0.$$

Now let's go along the path $x=y$: set $x=y$ to get $f(y,y) = \frac{y^2-y^3}{2y^2} = \frac{1}{2}(1-y)$ and take the limit as y goes to zero to get

$$\lim_{y\to 0} f(y,y) \;=\; \lim_{y\to 0}\frac{1}{2}(1-y) \;=\; \frac{1}{2}.$$

Since these limits are not equal, the limit **does not exist**. □

(4) *Solution.*

(a) We have $-x^2-y^2 = -(x^2+y^2)$, so this function is an upside-down paraboloid. For each c, we have that $-x^2-y^2=c$, so for each $c<0$, this equation gives us a circle with radius $\sqrt{-c}$. Thus, the level sets are all circles.

(b) For each c, the equation $\cos(x+y) = c$ only has a solution if $-1 \leq c \leq 1$. In this case, we have that $x + y = \cos^{-1}(c) + 2\pi k$ where k can be any integer (i.e. $k = \ldots, -2, -1, 0, 1, 2, 3, \ldots$). Solving this for y, we get that $y = -x + \cos^{-1}(c) + 2\pi k$. All that really matters for our purposes is that $\cos^{-1}(c) + 2\pi k$ is some number for each k, so each level set is a collection of lines with slope -1.

(c) As x gets big, the factor e^{-x^2} gets really close to zero really quickly, so the only place we'll see anything is close to the y-axis (where $x = 0$). Along this axis, we get the function y^2 and as we move away from this axis (as x gets bigger) we get the function $e^{-x^2}y^2$ where $0 < e^{-x^2} < 1$. □

(5) *Solution.*

(a) $\frac{\partial f}{\partial x} = 3x^2 + 2y + 0 = 3x^2 + 2y$ and $\frac{\partial f}{\partial y} = 0 + 2x + 1 = 2x + 1$.

(b) f decreases as x increases and f increases as x decreases, so $f_x(a, b) < 0$. f decreases as y increases and as y decreases, so $f_y(a, b) = 0$. If one looks at concavity in the x direction, we see that f is concave up, so $f_{xx}(a, b) > 0$. Another way to see this is to note that as x increases, f decreases more and more slowly.

(c) We first estimate $f_x(2, 1)$. Starting at $(2, 1)$, we have to go right to the point $(2.8, 1)$ to go from the level set $f = 0$ to the level set $f = 1$. We have to go left to the point $(1.4, 1)$ to go from the level set $f = 0$ to the level set $f = -1$. Thus, the change in f between these two points is 2 and the change in x is about $2.8 - 1.4 = 1.4$. This gives us that

$$f_x(2,1) \;\approx\; \frac{\Delta f}{\Delta x} \;=\; \frac{2}{1.4} \;\approx\; 1.43 \;\approx\; 1.5.$$

Likewise, we have to go up to the point $(2, 1.5)$ to go from the level set $f = 0$ to the level set $f = -1$. We have to go down to the point $(2, 0.5)$ to go from the level set $f = 0$ to the level set $f = 1$. Thus, the change in f between these two points is -2 and the change in 5 is about $1.5 - 0.5 = 1$. This gives us that

$$f_y(2,1) \;\approx\; \frac{\Delta f}{\Delta y} \;=\; \frac{-2}{1} \;=\; -2.$$

□

(6) *Solution.* First, we calculate the tangent plane to f at $(1, 2)$. This is the plane

$$z \;=\; f(1,2) + \frac{\partial f}{\partial x}(1,2)(x-1) + \frac{\partial f}{\partial y}(1,2)(y-2) \;=\; 5 - 2(x-1) + 3(y-2).$$

In order to approximate $f(1.1, 1.9)$, we plug this point into the equation of the tangent plane to get

$$f(1.1, 1.9) \;\approx\; 5 - 2(1.1-1) + 3(1.9-2) \;=\; 5 - 0.2 - 0.3 \;=\; 4.5.$$

□

(7) *Solution.*

(a)

$$\frac{\partial F}{\partial s} \;=\; \frac{\partial f}{\partial x}\big(x(s,t), y(s,t)\big)\frac{\partial x}{\partial s}(s,t) + \frac{\partial f}{\partial y}\big(x(s,t), y(s,t)\big)\frac{\partial y}{\partial s}(s,t).$$

(b) We have $x(2, 1) = 5$ and $y(2, 1) = 3$. We have $\frac{\partial x}{\partial s} = 2$, so $\frac{\partial x}{\partial s}(2, 1) = 2$. We also have $\frac{\partial y}{\partial s} = 2st$ so $\frac{\partial x}{\partial s}(2, 1) = 4$. Substituting all of this into the equation from part (a), we have

$$\begin{aligned}\frac{\partial F}{\partial s}(2,1) \;&=\; \frac{\partial f}{\partial x}\big(x(2,1), y(2,1)\big)\frac{\partial x}{\partial s}(2,1) + \frac{\partial f}{\partial y}\big(x(2,1), y(2,1)\big)\frac{\partial y}{\partial s}(2,1) \\ &=\; \frac{\partial f}{\partial x}(5,3)\cdot 2 + \frac{\partial f}{\partial y}(5,3)\cdot 4 \;=\; 3\cdot 2 + 5\cdot 4 \;=\; 26.\end{aligned}$$

□

(8) *Extra Credit Solution.* For $\mathbf{h} = (x, y)$, we have that $|\mathbf{h}| = \sqrt{x^2 + y^2}$. In particular, $|x| < |\mathbf{h}|$ and $|y| < |\mathbf{h}|$. Now we have $E(\mathbf{h}) = 2x + y^2$, we have

$$|E(\mathbf{h})| \;=\; |2x + y^2| \;\leq\; 2|x| + |y|^2 \;\leq\; 2|\mathbf{h}| + |\mathbf{h}|^2.$$

If $|\mathbf{h}| < 1$, then $|\mathbf{h}|^2 < |\mathbf{h}|$ and so $|E(\mathbf{h})| < 3|\mathbf{h}|$. Therefore, if we want $|E(\mathbf{h})|$ to be less than $0.01 = \frac{1}{100}$, we should choose $|\mathbf{h}|$ such that $3|\mathbf{h}| < \frac{1}{100}$. For example, we can choose $\delta = \frac{1}{300}$. $\square$