1. Give a vector $\mathbf{v}$ perpendicular to the plane that contains the line $x = 1 + t$, $y = 2 - 3t$, $z = 2 + 4t$ and the line $x = 2$, $y = -1 + 2t$, $z = 6 - t$. **(5 points)**

A vector pointing along the first line is $\mathbf{a} = \langle 1, -3, 4 \rangle$

A vector pointing along the second line is $\mathbf{b} = \langle 0, 2, -1 \rangle$

A normal vector to the plane containing both lines is then given by their cross product:

$$\mathbf{v} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -3 & 4 \\ 2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \mathbf{k} = -5\mathbf{i} + 1\mathbf{j} + 2\mathbf{k}$$

So $\mathbf{v} = \langle -5, 1, 2 \rangle$.

$$\boxed{\mathbf{v} = \langle -5, 1, 2 \rangle}$$

2. Find the angle θ between the planes $x = z + 8$ and $2y = 1 - 2x + z$. **(5 points)**

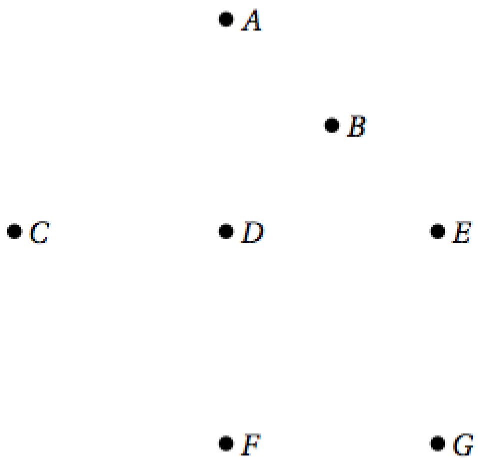
The angle θ between the two planes is equal to the angle between their normal vectors. Rewriting the equations of the planes as $x - z = 8$ and $2x + 2y - z = 1$, we see that their respective normal vectors are $\mathbf{n_1} = \langle 1, 0, -1 \rangle$ and $\mathbf{n_2} = \langle 2, 2, -1 \rangle$. Then θ satisfies

$$\cos(\theta) = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{|\mathbf{n_1}||\mathbf{n_2}|} = \frac{1(2) + 0(2) + (-1)(-1)}{\sqrt{1^2 + 0^2 + (-1)^2}\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{2+1}{\sqrt{2}\sqrt{9}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

It follows that $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ (this is equivalent to 45°).

$$\boxed{\theta = \frac{\pi}{4}}$$

3. Consider the points in the plane shown at right:



   (a) Circle the vector that is equal to $2\overrightarrow{DE} - \overrightarrow{CA}$.

   $\overrightarrow{DA}$ $\overrightarrow{DB}$ $\overrightarrow{DE}$ $\overrightarrow{DF}$ [✓ $\overrightarrow{DG}$]

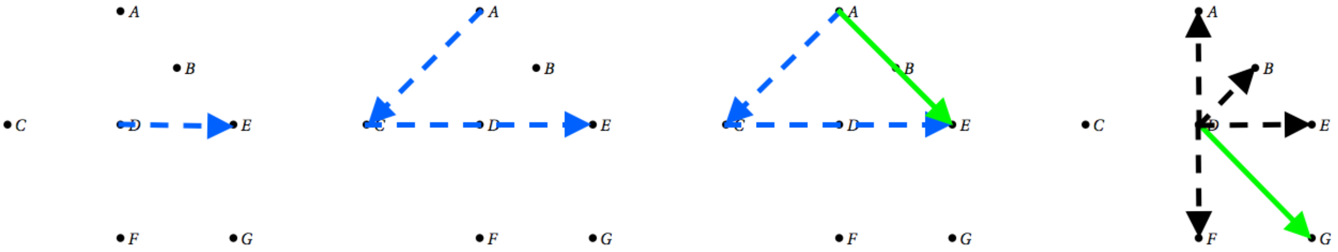
   (2 points)

   (b) Circle the vector that is equal to $\text{proj}_{\overrightarrow{CA}}\overrightarrow{DE}$.

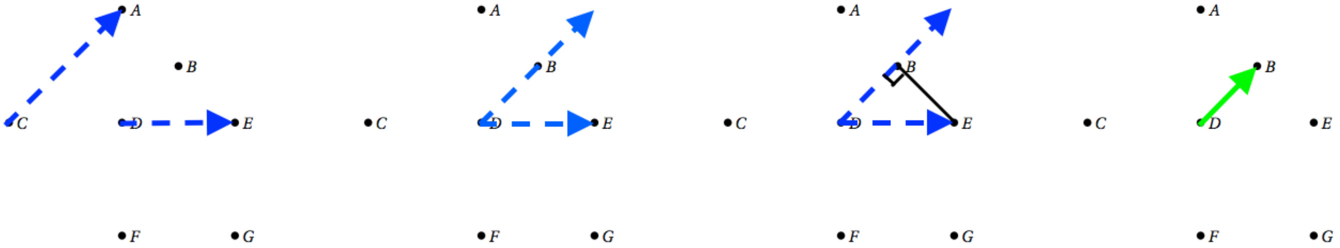
   $\overrightarrow{DA}$ [✓ $\overrightarrow{DB}$] $\overrightarrow{DE}$ $\overrightarrow{DF}$ $\overrightarrow{DG}$

   (2 points)

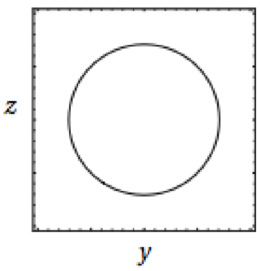
For part (a), we can represent $2\overrightarrow{DE}$ as $\overrightarrow{CE}$ and $-\overrightarrow{CA}$ as $\overrightarrow{AC}$. Then $2\overrightarrow{DE} - \overrightarrow{CA} = \overrightarrow{AE}$. Of the options given, the only one that has the same magnitude and goes in the same direction as $\overrightarrow{AE}$ is $\overrightarrow{DG}$.



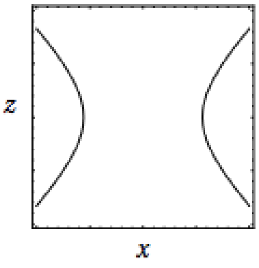
For part (b), $\text{proj}_{\overrightarrow{CA}}\overrightarrow{DE}$ is the component of $\overrightarrow{DE}$ along $\overrightarrow{CA}$, that is, the scalar multiple of $\overrightarrow{CA}$ closest to $\overrightarrow{DE}$. After drawing $\overrightarrow{CA}$ and $\overrightarrow{DE}$, we can move $\overrightarrow{CA}$ over so that it emanates from point D in order to better find the projection. Next, we draw a line from the point E to the copy of $\overrightarrow{CA}$ so that it forms a right angle, which will be at point B. Then $\overrightarrow{DB}$, portion of the copy of $\overrightarrow{CA}$ up to B, gives us $\text{proj}_{\overrightarrow{CA}}\overrightarrow{DE}$.



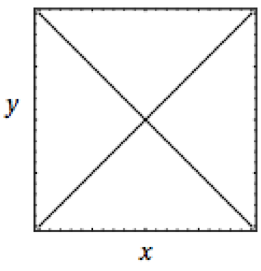
4. Suppose S is a quadric surface whose intersections with the following three planes are:



Plane {$x = 1$}

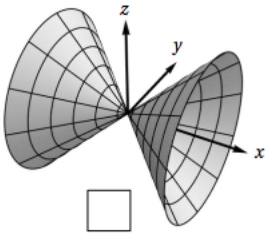


Plane {$y = -1.5$}



Plane {$z = 1$}

(a) Mark the box next to the picture of the portion of S where $-2 \leq x \leq 2$, $-2 \leq y \leq 2$, and $-2 \leq z \leq 2$:
(2 points)



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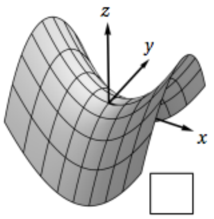

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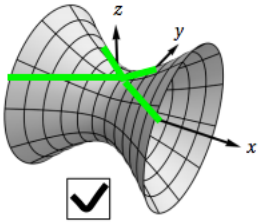

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Only the first and third pictures match view from the plane $x = 1$, that is, a circle in y and z. For the first picture, at $z = 1$, we would see a hyperbola in x and y, not a × shape, because when we freeze $z = 1$, there is no path from the right half of the figure to the left. Only the third picture matches all three of the plane intersections given.

(b) Circle the equation that S satisfies:

$x^2 + y^2 - z = 0$ [✓ $y^2 + z^2 - x^2 = 1$] $y^2 + z^2 = x^2$

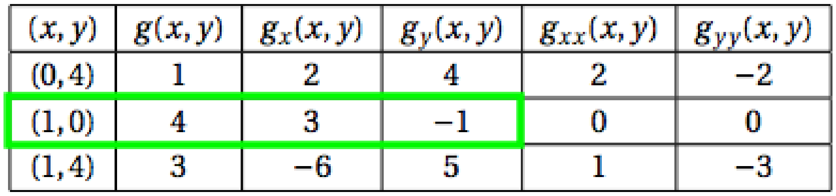
(2 points)
We know from the plane intersections that if we set $x = 1$, we should get a circle in y and z. When we set $x = 1$ in the first choice, we get $1 + y^2 - z = 0 \implies 1 = z - y^2$, and is not a circle; when we set $x = 1$ in the second choice, we get $y^2 + z^2 - 1 = 1 \implies y^2 + z^2 = 2$, and is a circle; and when we set $x = 1$ in the third choice, we get $y^2 + z^2 = 1$, which is also a circle. Then we can rule out the first choice. Next, if we set $z = 1$, we should get two lines in x and y. When we set $z = 1$ in the second choice, we get $y^2 + 1 - x^2 = 1 \implies y^2 = x^2$, and this forms two lines, $y = x$ and $y = -x$. When we set $z = 1$ in the third choice, we get $y^2 + 1 = x^2 \implies x^2 - y^2 = 1$, which is a hyperbola and not two lines, so we may rule out the third choice as well. Then the answer must be the second choice. Alternatively, the second choice is the only one that gives the equation of a hyperboloid of one sheet, matching the rightmost picture in part (a). Alternatively, the first and third choices go through the point $(0, 0, 0)$, and we can see in the picture from part (a) that S does not.

5. Consider the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^4\cos^2(x)}{x^4+y^4}$. Does this limit exist? If so, what is its value? Justify your answer. **(5 points)**

Set $f(x,y) = \dfrac{y^4\cos^2(x)}{x^4+y^4}$. Then f is not defined at $(0,0)$, since we get the indeterminate form $\frac{0}{0}$. Instead, we approach $(0,0)$ along different paths. The most basic choices of path are moving along the x-axis (where $y=0$) and moving along the y-axis (where $x=0$). Along the x-axis, we have $f(x,0) = \dfrac{(0^4)\cos^2(x)}{x^4+0^4} = \dfrac{0}{x^4} = 0$ for $x \neq 0$, so the limit of the function as we approach along the x-axis is 0. Along the y-axis, we have $f(0,y) = \dfrac{y^4\cos^2(0)}{0^4+y^4} = \dfrac{y^4}{y^4} = 1$ for $y \neq 0$, so the limit of the function as we approach along the y-axis is 1. Since f approaches different values depending on how we approach $(0,0)$, that is, how $(x,y)\to(0,0)$, we conclude that the limit does not exist.

(If we had not been able to find different paths that give different limits, we might try proving that the limit exists using the ϵ, δ definition or polar coordinates.)

6. Find the equation of the tangent plane to the graph of g at the point $(1,0,4)$ basd on the data in the table at right. **(4 points)**



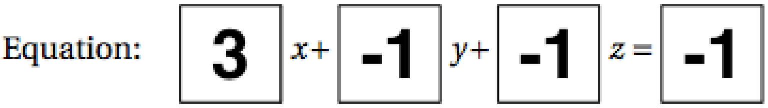
(x,y)	$g(x,y)$	$g_x(x,y)$	$g_y(x,y)$	$g_{xx}(x,y)$	$g_{yy}(x,y)$
$(0,4)$	1	2	4	2	-2
$(1,0)$	4	3	-1	0	0
$(1,4)$	3	-6	5	1	-3

The general formula for the tangent plane to a function f at a point (a,b) is given by

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

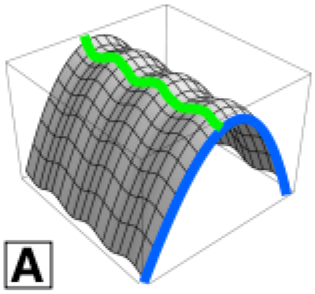
For our function g, we see from the table that $g(1,0)=4$, $\frac{\partial g}{\partial x}(1,0) = g_x(1,0) = 3$, and $\frac{\partial g}{\partial y}(1,0) = g_y(1,0) = -1$. Then the equation for the tangent plane to the graph of g at the point $(1,0,4)$ is given by

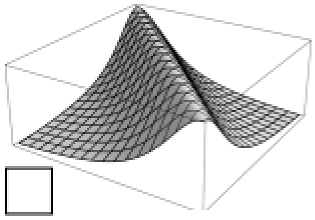
$$\begin{aligned} & z-4 = 3(x-1)+(-1)(y-0) \\ \Longrightarrow\ & z-4 = 3x-3-y \\ \Longrightarrow\ & -4+3 = 3x-y-z \\ \Longrightarrow\ & -1 = 3x+(-1)y+(-1)z \end{aligned}$$

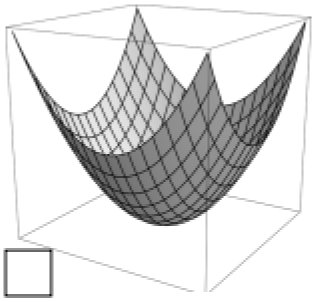


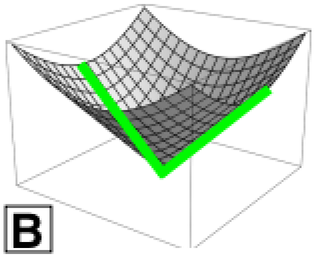
Equation: $\boxed{3}\,x + \boxed{-1}\,y + \boxed{-1}\,z = \boxed{-1}$

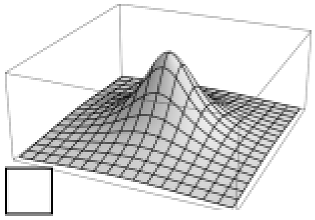
7. For each function label its graph from among the options below: **(2 points each)**
(A) $-y^2 + \sin(x)$ (B) $\sqrt{x^2 + y^2}$

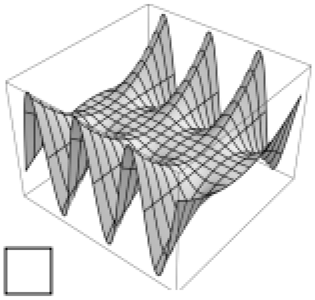








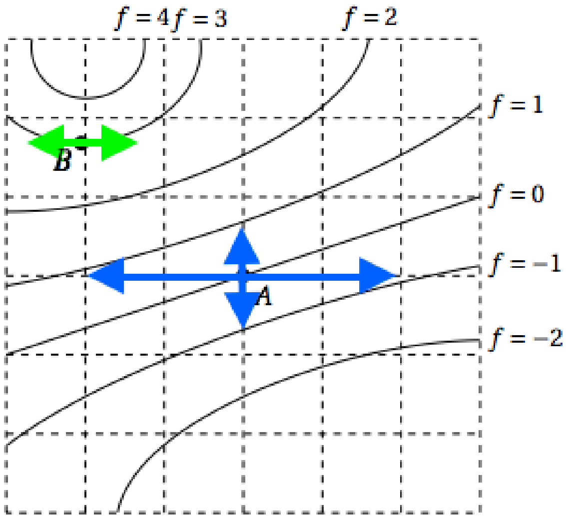




For (A), if we set $x = c$ for c a constant, we get $z = -y^2 + \sin(c)$ where $\sin(c)$ is a constant, so in one direction, we should have a downward-facing parabola. If we set $y = d$ for d a constant, we get $z = -d^2 + \sin(x)$ where $-d^2$ is a constant, so in the other direction, we should have a sine function. The upper left-hand graph is the only one that satisfies both of these.

For (B), if we set $x = c$ for c a general constant, we get $z = \sqrt{c^2 + y^2} \leftrightarrow z^2 - y^z = c^2$, which forms a hyperbola in y and z. This matches both the upper right-hand graph and the lower left-hand graph. If we set $x = 0$ specifically, we get $z = \sqrt{0^2 + y^2}$, so $z = y$ or $z = -y$. Even without having the axes labelled, this general shape only occurs in the lower left-hand graph. Alternatively, $z = \sqrt{x^2 + y^2}$ is the positive half of $z^2 = x^2 + y^2$, which is the equation of a double cone, and the lower left-hand graph is the only one that looks like half of a double cone.

8. Consider the countour plot of $f(x, y)$ shown at right, where the dotted grid is made of unit squares.



   (a) At the point B, the derivative $\frac{\partial f}{\partial x}$ is:

   negative [✓ zero] positive

   (1 point)

   (b) Estimate the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point A. For each, circle the number below that is closest to your estimate.
   (3 points)

$\frac{\partial f}{\partial x}$: −3 −2.5 −2 −1.5 −1 [✓ −0.5] 0 0.5 1 1.5 2 2.5 3

$\frac{\partial f}{\partial y}$: −3 −2.5 −2 −1.5 −1 −0.5 0 0.5 1 [✓ 1.5] 2 2.5 3

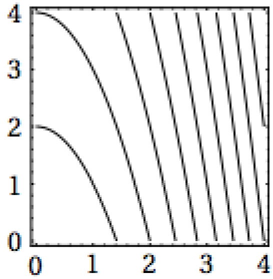
For part (a), recall that $\frac{\partial f}{\partial x}$ at B is the rate at which f changes if we start at B and vary x (while holding y constant). If we move a small amount to the left or to the right from point B, we're still on (or very nearly on) the contour where $f = 3$, so f does not change if we start at B and vary x. Then $\frac{\partial f}{\partial x} = 0$.

For part (b), consider $\frac{\partial f}{\partial x}$ at point A first. If we hold y constant and vary x, we immediately come off the contour $f = 0$, so we expect $\frac{\partial f}{\partial x} \neq 0$. To find an approximate value, we consider how much we need to vary x in order to reach a new contour. To get to the contour at $f = -1$, we need to move about two units to the right, so in order to decrease f by 1, we need to increase x by 2, which suggests $\frac{\partial f}{\partial x} = \frac{-1}{2} = -0.5$. To get to the contour at $f = 1$, we need to move about two units to the left, so in order to increase f by 1, we need to decrease x by 2, which confirms $\frac{\partial f}{\partial x} = \frac{1}{-2} = -0.5$.

Now we consider $\frac{\partial f}{\partial y}$ at point A. If we hold x constant and vary y, we immediately come off the contour $f = 0$, so we expect $\frac{\partial f}{\partial y} \neq 0$. To find an approximate value, we consider how much we need to vary y in order to reach a new contour. To get to the contour at $f = -1$, we need to move about $\frac{2}{3}$ of a unit down, so in order to decrease f by 1, we need to decrease y by $\frac{2}{3}$, which suggests $\frac{\partial f}{\partial y} = \frac{-1}{-\frac{2}{3}} = \frac{3}{2} = 1.5$. To get to the contour at $f = 1$, we need to move about $\frac{2}{3}$ of a unit up, so in order to increase f by 1, we need to increase y by $\frac{2}{3}$, which confirms $\frac{\partial f}{\partial y} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1.5$.

9. Suppose the temperature, measured in °C, on a tabletop is given by $F(x, y) = 40 - x^2 - y$, where x and y have units of centimeters.

   (a) Mark the box below the contour plot that best matches the function F. **(1 point)**



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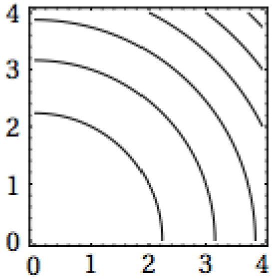

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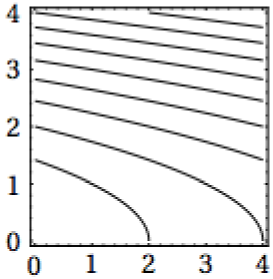

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   (b) Suppose a small bug positioned at $(3, 1)$ is travelling so that its x-coordinate is **decreasing** at a rate of 1 cm/s and its y-coordinate is **increasing** at a rate of 2cm/s. Use the Chain Rule to calculate the rate at which the temperature is changing from the bug's perspective. **(5 points)**

For part (a), each contour represents a curve given when we fix $F(x, y) = C$ for C some constant, so each contour is of the form $C = 40 - x^2 - y$, which rearranges to $y = -x^2 + (40 - C)$. Then every contour should have the shape of a vertical, downward-opening parabola, which only matches the leftmost plot (the middle plot has circles instead of parabolas, and the rightmost plot has horizontal parabolas). Alternatively, x^2 tends to change faster than y^1, so we expect to see contours bunching more closely together in the x-direction than they do in the y-direction, which again only matches the leftmost plot.

For part (b), let the bug's position be given by $(x(t), y(t))$, where $(x(0), y(0)) = (3, 1)$. The rate at which the temperature is changing from the bug's perspective at time $t = 0$ is given by $\frac{dF}{dt}(0)$, and by the Chain Rule, $\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$. Since $F(x, y) = 40 - x^2 - y$, we know that $\frac{\partial F}{\partial x} = -2x$ and $\frac{\partial F}{\partial y} = -1$, and it is given that the bug's x-coordinate is decreasing at a rate of 1 cm/s, so $\frac{dx}{dt}(0) = -1$, and its y-coordinate is increasing at a rate of 2cm/s, so $\frac{dy}{dt}(0) = 2$.

Then

$$\begin{aligned}
\frac{dF}{dt}(0) &= \frac{\partial F}{\partial x}(x(0), y(0))\frac{dx}{dt}(0) + \frac{\partial F}{\partial y}(x(0), y(0))\frac{dy}{dt}(0) \\
&= \frac{\partial F}{\partial x}\Big(3, 1\Big)(-1) + \frac{\partial F}{\partial y}\Big(3, 1\Big)(2) \\
&= (-2)3(-1) + (-1)(2) \\
&= 6 - 2 \\
&= 4
\end{aligned}$$

So from the bug's perspective, at the point $(3, 1)$, the temperature is changing at a rate of 4° C per second.

rate = 4 °C/s