1. Suppose A, B, and C are three distinct points in $\mathbb{R}^3$ with $\mathbf{v} = \overrightarrow{AB} = \langle 3,\ 0,\ 1\rangle$ and $\mathbf{w} = \overrightarrow{AC} = \langle 2,\ 1,\ 0\rangle$.

$\boxed{\text{2024 Fall}}$

(a) Find a normal vector $\mathbf{n}$ to the plane P containing A, B, C. **(2 points)**

Take $\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} \vec{k}$

$= -1\vec{i} + 2\vec{j} + 3\vec{k}$

$\mathbf{n} = \langle -1,\ 2,\ 3 \rangle$

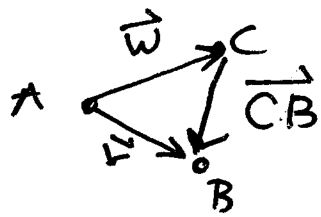
(b) Find the area of the triangle with vertices A, B, C. **(2 points)**

$\text{Area} = \frac{1}{2} \text{Area} \left(\text{parallelogram spanned by } w, v \right) = \frac{1}{2} |\vec{n}|$

$= \frac{1}{2} \sqrt{1^2 + 2^2 + 3^2} = \frac{1}{2}\sqrt{14}$

Area $= \frac{1}{2}\sqrt{14}$

(c) Compute $\overrightarrow{CB}$. **(1 point)**



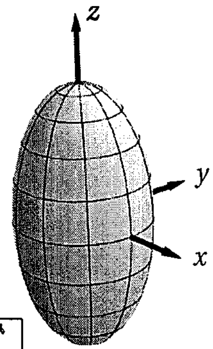
$\overrightarrow{CB} = \vec{v} - \vec{w}$

$= \langle 3,0,1\rangle - \langle 2,1,0\rangle = \langle 1,-1,1\rangle$

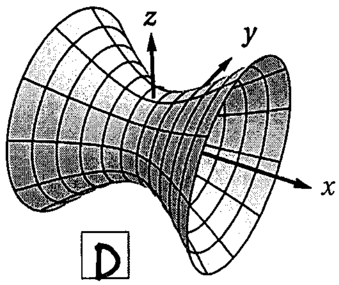
$\overrightarrow{CB} = \langle 1,\ -1,\ 1 \rangle$

Scratch Space

2. For each quadric surface below, find the corresponding equation from the options at right and write the corresponding letter in the box next to the graph. **(2 points each)**



C



D

(A) $x^2 + y^2 + 4z^2 - 1 = 0$

(B) $x^2 + y^2 - z^2 - 1 = 0$

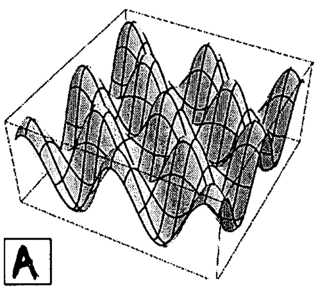
(C) $4x^2 + 4y^2 + z^2 - 1 = 0$

(D) $x^2 - y^2 - z^2 + 1 = 0$

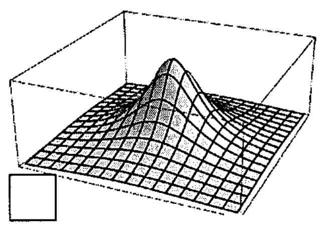
3. For each function, label its graph from among the options below by writing the corresponding letter in the box next to the graph. **(2 points each)**

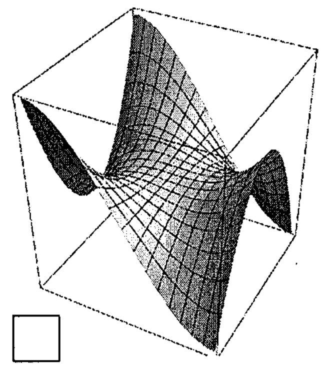
(A) $\cos(x)\cos(y)$

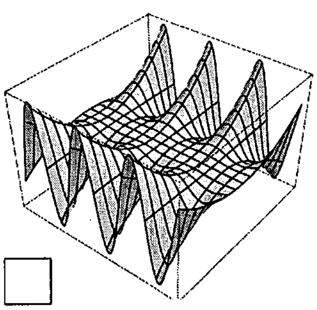
(B) $\dfrac{1}{1+(x+y)^2}$

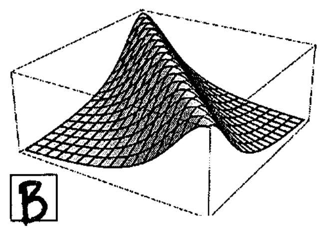


A

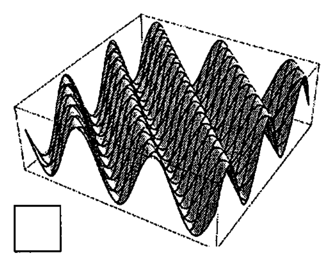








B



Scratch Space

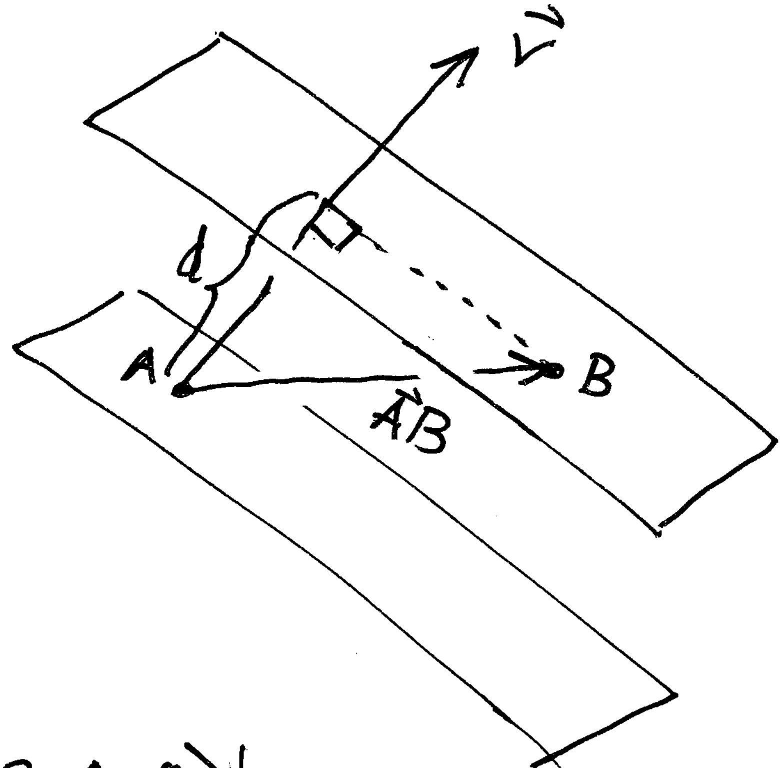
4. Consider the planes $V = \{x - y + 2z = 2\}$ and $W = \{2x - 2y + 4z = 8\}$ in $\mathbb{R}^3$.

(a) The two planes V and W are parallel. Explain why. **(1 point)**

From the equations, normals to the planes are $\vec{v} = \langle 1, -1, 2\rangle$ and $\vec{w} = \langle 2, -2, 4\rangle$. As $\vec{w} = 2\vec{v}$, these have the same direction and so the planes are parallel.

(b) Find a point A on V and a point B on W. Compute the distance between V and W by using a vector projection involving the vector $\overrightarrow{AB}$. **(4 points)**

Take $A = (2,0,0)$ and $B = (4,0,0)$ so $\overrightarrow{AB} = \langle 2,0,0\rangle$.



Then

$$d = \left|\operatorname{proj}_{\vec{v}} \overrightarrow{AB}\right|$$

$$= \frac{|\vec{v} \cdot \overrightarrow{AB}|}{|\vec{v}|}$$

$$= \frac{|\langle 1,-1,2\rangle \cdot \langle 2,0,0\rangle|}{\sqrt{1^2 + (-1)^2 + 2^2}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}.$$

distance = $\sqrt{2/3}$

5. Let $f(x,y) = \dfrac{xy}{x^2+2y^2}$ for $(x,y) \neq 0$. Does the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, what is its value? Justify your answer. **(5 points)**

Along x-axis, have $f(x,0) = \dfrac{0 \cdot y}{0+2y^2} = 0$ for $x \neq 0$.

Along $y=x$, have $f(x,x) = \dfrac{x^2}{x^2+2x^2} = \dfrac{x^2}{3x^2}$

$= 1/3$ for $x \neq 0$.

As these differ, the limit does not exist

$\lim_{(x,y)\to(0,0)} f(x,y) =$ DNE

Scratch Space

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = 2+\sqrt{x^2+y^2}$. $= 2+(x^2+y^2)^{+1/2}$

(a) The function f is differentiable at $(1,0)$. Find the equation for the tangent plane to its graph at the point $(1,0,3)$. **(4 points)**

$$f_x = 0 + \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2x) = \frac{x}{\sqrt{x^2+y^2}} \qquad f(1,0) = 2+\sqrt{1^2} = 3$$

$$f_y = 0 + \frac{1}{2}(x^2+y^2)^{-1/2}\cdot(2y) = y/\sqrt{x^2+y^2}$$

Tangent plane is

$$z = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-1)$$
$$= 3 + \frac{1}{\sqrt{1^2+0}}(x-1) + 0\cdot(y-1)$$
$$= 3 + x - 1 = x+2$$

$$\Rightarrow -x+z=2$$

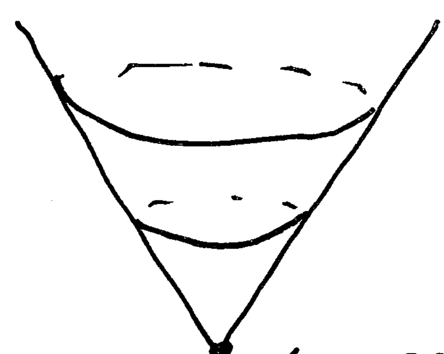
Equation: $\boxed{-1}\,x + \boxed{0}\,y + \boxed{1}\,z = \boxed{2}$

(b) At the point $(0,0)$, the function f is (circle your answer): **(1 point)**

continuous	differentiable	both	neither

Scratch Space

Graph of f



← pointy!

7. Let $f(x,y)$ be a function with values and derivatives in the table. Use linear approximation to estimate $f(3.1, 5.9)$. **(3 points)**

(x,y)	$f(x,y)$	$\frac{\partial f}{\partial x}(x,y)$	$\frac{\partial f}{\partial y}(x,y)$
$(-1,3)$	0	4	4
$(2,1)$	2	-1	3
$(2,4)$	3	7	7
$(3,6)$	1	-3	5

Approximate at $(3,6)$

$f(3.1, 5.9)$

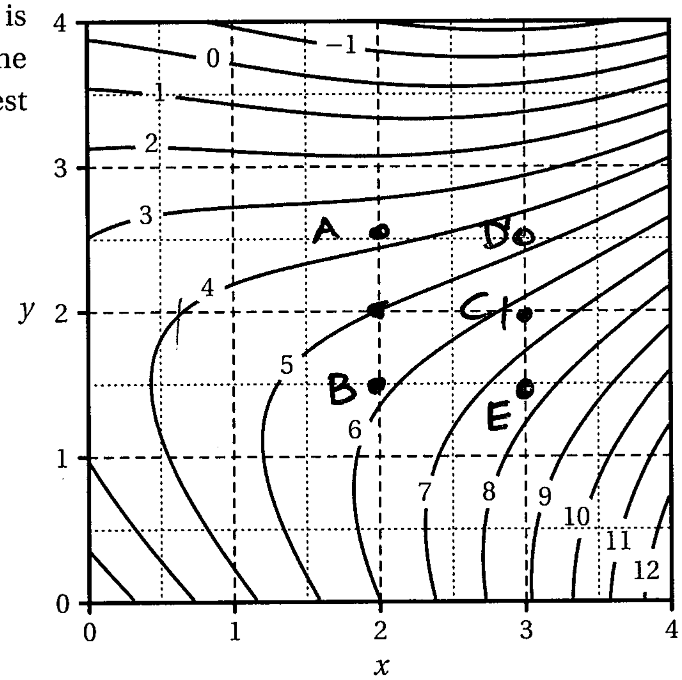
$\approx f(3,6) + f_x(3,6)\Delta x + f_y(3,6)\Delta y$

$= 1 + (-3)(0.1) + 5 \cdot (-0.1)$

$= 1 - 0.3 - 0.5 = 0.2$

$f(3.1, 5.9) \approx$ 0.2

8. The contour map of a differentiable function $g(x,y)$ is shown at right, where each level curve is labeled by the corresponding value of g. For each part, circle the best possible answer. **(1 point each)**



(a) $\frac{\partial g}{\partial y}(2,2)$ is

-4	-2	-1	0	1	2	4

(b) $\frac{\partial g}{\partial x}(2,2)$ is — negative — zero — positive

(c) $\frac{\partial^2 g}{\partial x^2}(2,2)$ is — negative — zero — positive

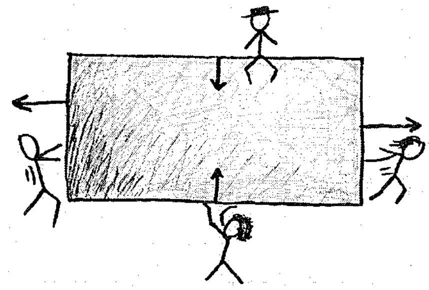
(d) $\frac{\partial^2 g}{\partial y \partial x}(2,2)$ is — negative — zero — positive

$g_y(2,2) \approx \frac{g(A) - g(B)}{\Delta y} = \frac{3.75 - 5.75}{1} = -2$

$g_y(C = (3,2)) \approx \frac{g(D) - g(E)}{\Delta y} = \frac{4.6 - 7.5}{1} = -2.9$

So $g_y \searrow$ as x increases.

9. A gelatinous rectangle has height $h = 10m$ and width $w = 20m$ at time $t = 0$. Suppose the rectangle is compressed vertically so the height **decreases** at a rate of $2m/s$ and is expanded horizontally so that the width **increases** at a rate of $1m/s$, Assuming it keeps a rectangular shape throughout, let $A(t)$ denote its area at time t. Use the Chain Rule to compute $\frac{dA}{dt}(0)$. **(5 points)**



$$A = hw$$

$$\frac{dA}{dt} = \overset{w}{\overset{\parallel}{\frac{\partial A}{\partial h}}} \frac{dh}{dt} + \overset{h}{\overset{\parallel}{\frac{\partial A}{\partial w}}} \frac{dw}{dt}$$

So $$\frac{dA}{dt}(0) = (20\,m)\cdot(-2\,m/s) + (10\,m)\cdot(1\,m/s)$$
$$= -30\,m^2/s$$

$\frac{dA}{dt}(0) =$ -30 m^2/s

Scratch Space