1. Consider the function $f = x^3 + y^3 + 3xy$.

local

(a) It turns out the critical points of f are $(0,0)$ and $(-1,-1)$. Classify them into mins, maxes, and saddles. **(4 points)**

$f_x = 3x^2 + 3y$ $\qquad$ $f_y = 3y^2 + 3x$

$\boxed{f_{xy} = 3 = f_{yx}}$

$f_{xx} = 6x$ $\qquad$ $f_{yy} = 6y$

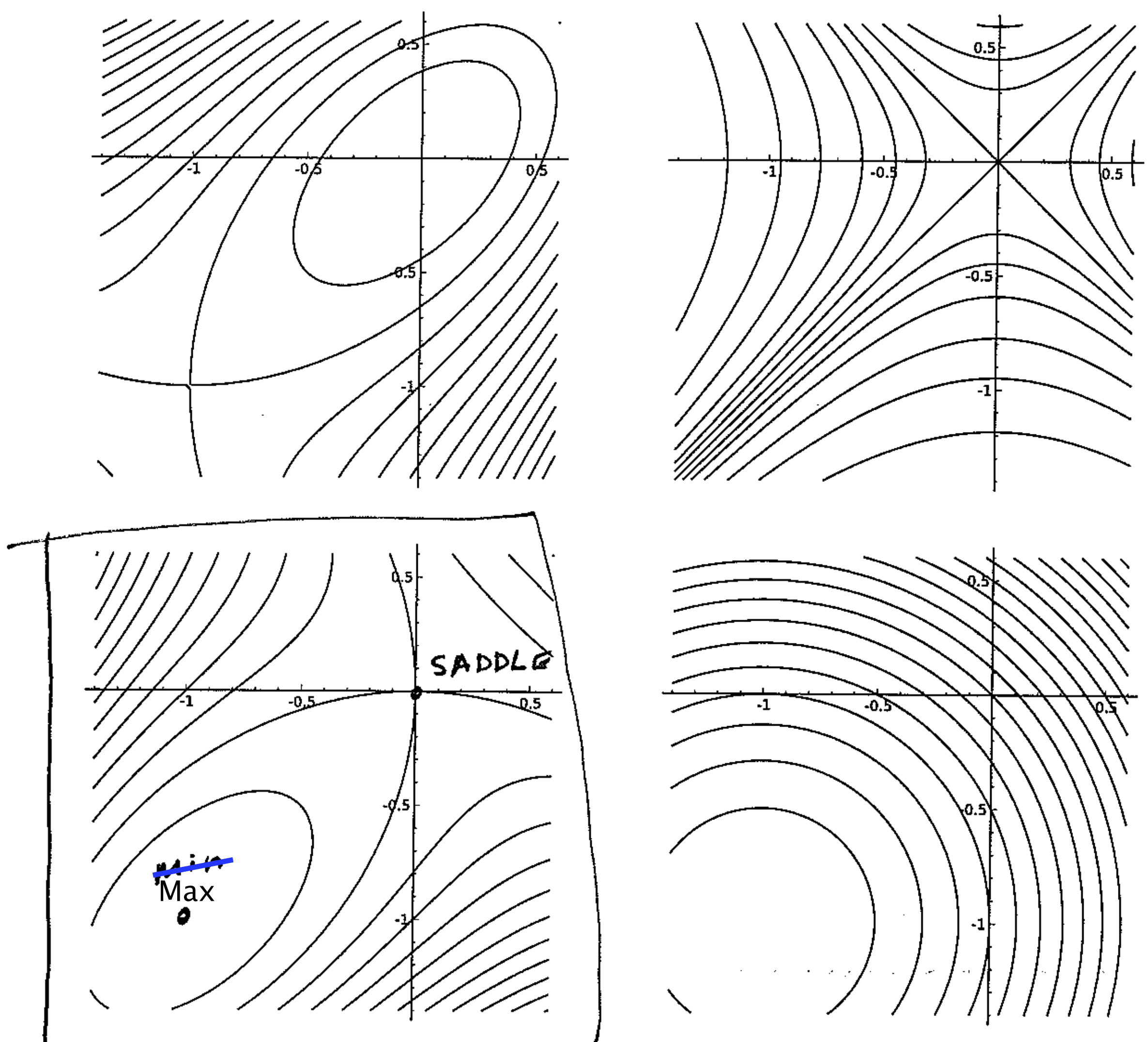
At $(0,0)$, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} = -9 < 0 \Rightarrow$ SADDLE

At $(-1,-1)$, $D = \begin{vmatrix} -6 & 3 \\ 3 & -6 \end{vmatrix} = 36 - 9 = 27 > 0$ and

$f_{xx} = -6 < 0$

$\Rightarrow$ LOCAL MAX

(b) Based on your answer in (a), circle the correct contour diagram of f. **(1 point)**



2. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 - 2x + y^2 - 2y$.

(a) Use Lagrange multipliers to find the max and min of f on the circle $x^2+y^2 = 8$. **(6 points)**

Take $g(x,y) = x^2+y^2$, so constraint is $g = 8$. Consider

$$\nabla f = (2x-2,\ 2y-2) = \lambda \nabla g = \lambda(2x,\ 2y).$$

This gives: $2(x-1) = 2\lambda x$ and $2(y-1) = 2\lambda y$.

Solving for λ gives: $1 - \frac{1}{x} = \lambda = 1 - \frac{1}{y} \Rightarrow -\frac{1}{x} = -\frac{1}{y}$

$\Rightarrow x = y$. Since $g = 8$, this means $2x^2 = 8 \Rightarrow x = y = \pm 2$.

<u>Critical Points:</u>

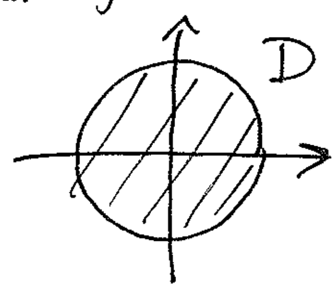
$(2,2)$ has $f = 2^2 - 2\cdot 2 + 2^2 - 2\cdot 2 = 0$ $\boxed{\text{MIN}}$

$(-2,-2)$ has $f = (-2)^2 - 2(-2) + (-2)^2 - 2(-2) = 16$ $\boxed{\text{MAX}}$

(b) Consider the region D where $x^2 + y^2 \le 8$. Explain why f must have a global min and max on D. **(2 points)** The region D is closed and bounded.

Since f is continuous (its just a polynomial) the Extreme Value Theorem gaurentees there are global extrema.

(c) Find the global min and max of f on D. **(3 points)**



The global exterma occur either on the boundary circle <u>or</u> at a pt where $\nabla f = \vec{0}$.

$$\nabla f = (2x-2,\ 2y-2) = (0,0) \Longrightarrow x = y = 1.$$

Combining with (a) we have 3 critical pts:

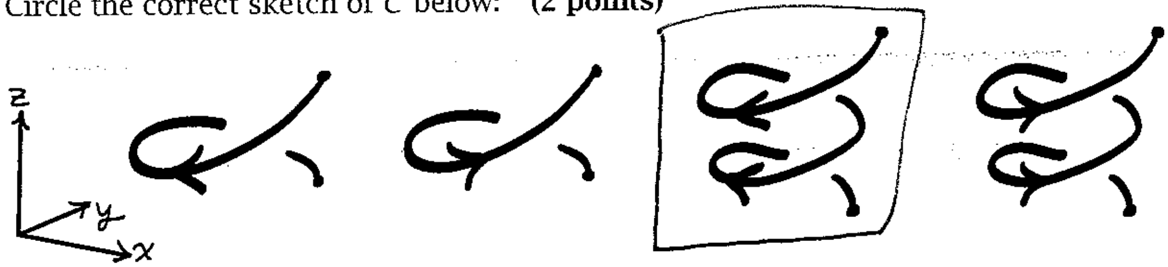
$(1,1)$ with $f = -2$ $\boxed{\text{MIN}}$ $(2,2)$ with $f = 0$ $\boxed{\text{NEITHER}}$ $(-2,-2)$ with $f = 16$ $\boxed{\text{MAX}}$

3. Let C be the portion of a helix parameterized by

$$\mathbf{r}(t) = (\cos(2t), -\sin(2t), 9-t) \quad \text{for } 0 \le t \le 2\pi.$$

The curve starts at (1,0,9) and goes down (decreasing z coordinate). It also has to spiral around twice.

(a) Circle the correct sketch of C below: **(2 points)**



(b) Compute the length of C. **(5 points)**

$$\text{Length} = \int_C 1\, ds = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{5}\, dt = \sqrt{5}\,t\Big|_{t=0}^{2\pi}$$

$$\boxed{= 2\pi\sqrt{5}.}$$

$$\vec{r}\,'(t) = (-2\sin 2t, -2\cos 2t, -1)$$

$$|\vec{r}\,'(t)| = \sqrt{(-2\sin 2t)^2 + (-2\cos 2t)^2 + (-1)^2}$$

$$= \sqrt{4\sin^2 2t + 4\cos^2 2t + 1}$$

$$= \sqrt{5}$$

(c) Suppose C is made of material with density given by $\rho(x,y,z) = x + z$. Give a line integral for the mass of C, and reduce it to an ordinary definite integral (something like $\int_0^1 t^2 \sin t\ dt$). **(3 points)**

$$\text{Mass} = \int_C \rho\, ds = \int_C x + z\, ds = \int_0^{2\pi} (\cos(2t) + 9 - t)\,\underbrace{\sqrt{5}\, dt}$$

$$= |\vec{r}\,'(t)|\, dt$$
$$= ds.$$

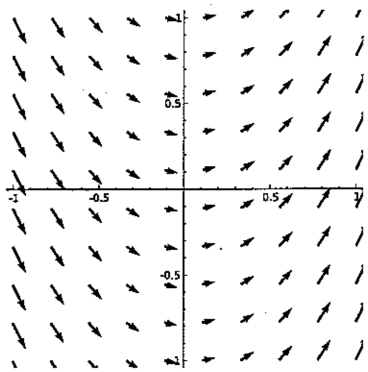
4. Let C be the curve parameterized by $\mathbf{r}(t) = (e^t, t)$ for $0 \le t \le 1$, and consider the vector field $\mathbf{F} = (1, 2y)$.

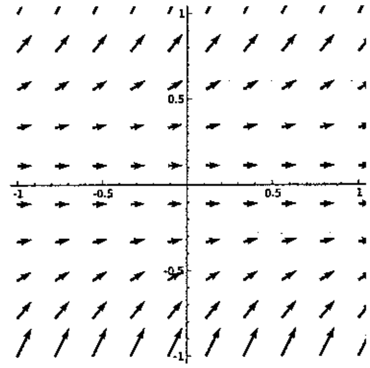
(a) Circle the picture of $\mathbf{F}$ below: **(2 points)**

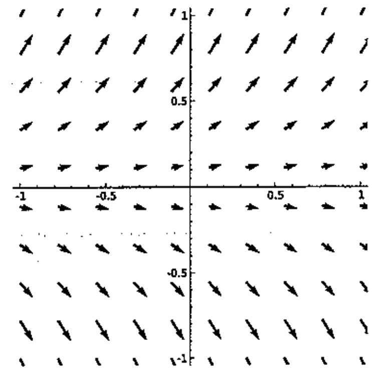
F = ↗ if y>0

F = → if y=0

F = ↘ if y<0







(b) Directly compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(5 points)**

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (1, 2t) \cdot (e^t, 1)\, dt$$

$$= \int_0^1 e^t + 2t\, dt = e^t + t^2 \Big|_{t=0}^{1} = e^1 - e^0 + 1 - 0$$

$$= e.$$

(c) The vector field $\mathbf{F}$ is conservative. Find $f\colon \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \mathbf{F}$. **(2 points)**

$$f = \int \frac{\partial f}{\partial x}\, dx = \int 1\, dx = x + C(y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial C}{\partial y} = 2y \Rightarrow C = y^2.$$

Thus $\boxed{f = x + y^2}$

<u>Check</u>:

$\nabla f = (1, 2y)$ ✓

(d) Use your answer in (c) to check your answer in (b). **(2 points)**

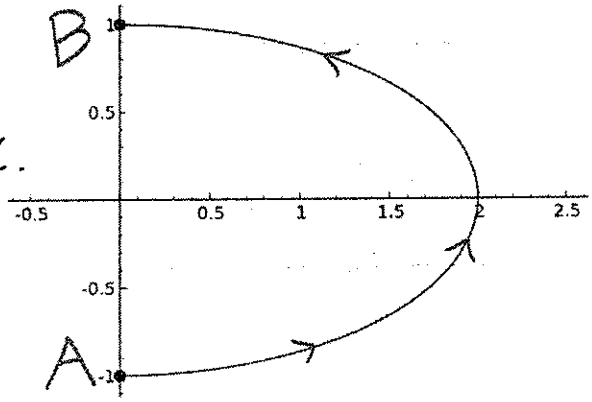
By the Fund Thm of Line Ints:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) = f(e,1) - f(1,0)$$

$$= (e + 1^2) - (1 + 0^2) = e$$ ✓

5. Let C be the indicated portion of the ellipse $\frac{x^2}{4} + y^2 = 1$ between $A = (0, -1)$ and $B = (0, 1)$.

(a) Give a parameterization $\mathbf{r}$ of C, indicating the domain so that it traces out precisely the segment indicated. **(3 points)**



Use y as the parameter, and then express y in terms of x.

$$\frac{x^2}{4} + y^2 = 1 \Rightarrow x^2 = 4(1-y^2)$$

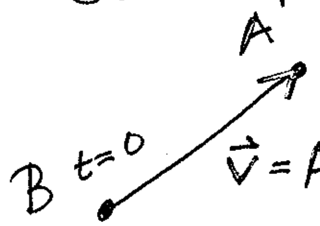
$$\Rightarrow x = +2\sqrt{1-y^2}$$

from picture

So $\vec{r}(t) = (2\sqrt{1-t^2}, t)$ for $-1 \leq t \leq 1$

(b) Let L be the line segment joining B to A. Give a parameterization $\mathbf{f}\colon [0,1] \to \mathbb{R}^2$ of L so that $\mathbf{f}(0) = B$ and $\mathbf{f}(1) = A$. **(2 points)**

General form:



$\vec{v} = A - B$

$\vec{f}(t) = B + t\vec{v} = B + t(A-B) = tA + (1-t)B$

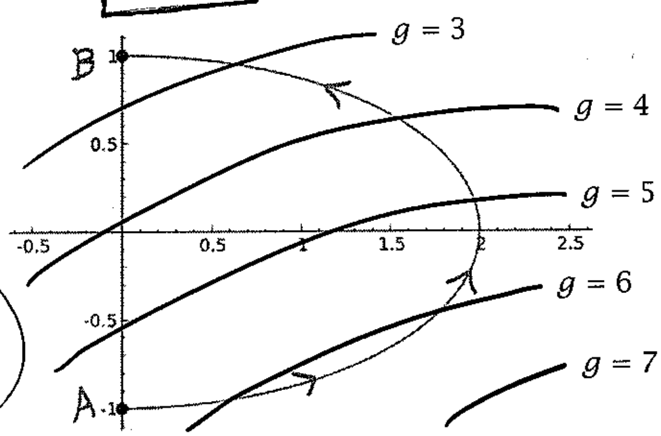
Specific case:

$\vec{f}(t) = t(0,-1) + (1-t)(0,1)$

$= (0, 1-2t)$ for $0 \leq t \leq 1$.

(c) Suppose $g\colon \mathbb{R}^2 \to \mathbb{R}$ is a function whose level sets are indicated below. Circle the sign of $\int_C g\, ds$ **(1 point)**

$\boxed{\text{positive}}$ negative 0



Reason:

$$\int_C g\, ds = \underbrace{\text{Length}(C)}_{>0} \cdot \underbrace{\left(\text{Average of } g \text{ on } C\right)}_{\approx 4.5}$$