1. Suppose $f(x,y)$ has values and partial derivatives as in the table at right.

(a) Find all the critical points you can from the given data and classify them into local mins, local maxes, and saddles. **(3 points)**

(x,y)	f	f_x	f_y	f_{xx}	f_{xy}	f_{yy}
(1,0)	2	0	1	1	0	2
(0,1)	3	0	0	−3	1	−2
(2,0)	0	−1	1	1	0	2
(1,2)	5	0	0	−1	3	2

Crit pts are where $\nabla f = (f_x, f_y) = \vec{0}$.

$(0,1)$: $D = \begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix} = 5 > 0$

and $f_{xx} = -3 < 0 \Rightarrow$ Local max.

$(1,2)$: $D = \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} = -11 < 0$

$\Rightarrow$ Saddle

Local mins (if any): None

Local maxes (if any): $(0,1)$

Saddles (if any): $(1,2)$

(b) Let $L(x,y)$ denote the linear approximation to $f(x,y)$ at the point $(0,1)$. Is $L(0.1, 1.1)$ likely to be larger than, equal to, or smaller than $f(0.1, 1.1)$? Circle your answer below **(1 point)**

$L(0.1, 1.1) > f(0.1, 1.1)$	$L(0.1, 1.1) = f(0.1, 1.1)$	$L(0.1, 1.1) < f(0.1, 1.1)$

f has a local max at (0,1), so tangent plane (which is the graph of L) is horizontal and z=f(x,y) lies below z=L(x,y)

2. Find an equation of the tangent plane to the surface defined by the equation $\underbrace{\frac{1}{2}x^2 + xy + 3yz^2}_{f} = 7$ at the point $(2,1,1)$. **(3 points)**

Normal vector is $\nabla f(2,1,1) = \langle 3, 5, 6 \rangle$

since $\nabla f = \langle x+y, x+3z^2, 6yz \rangle$. So eqn is:

$3(x-2) + 5(y-1) + 6(z-1) = 0$

$\Rightarrow 3x + 5y + 6z = 17$

Equation: $\boxed{3}\,x + \boxed{5}\,y + \boxed{6}\,z = \boxed{17}$

3. A tiny spaceship is orbiting on the path given by $\underbrace{x^2+y^2=4}_{g(x,y)}$. The solar radiation at a point (x,y) in the plane is $f(x,y)=(x+2)y=xy+2y$.

(a) Use the method of Lagrange multipliers to find the maximum value M and minimum value m of solar radiation experienced by the tiny spaceship in its orbit. **(5 points)**

$$\nabla f=\langle y, x+2\rangle=\lambda\nabla g=\lambda\langle 2x,2y\rangle$$

$$\Rightarrow y=\lambda 2x \text{ and } x+2=\lambda 2y.$$

By 1st eqn, if $x=0$ so does y, but $(0,0)$ is not on the given curve; so we can assume $x\neq 0$ and solve for λ to get $2\lambda=y/x$. Combining with the 2nd equation gives

$x+2=y^2/x \Rightarrow x^2+2x=y^2$. Using $x^2+y^2=4$ this implies $2x^2+2x=4 \Rightarrow x^2+x-2=0 \Rightarrow x=\frac{-1\pm\sqrt{1^2+8}}{2}$ $=\frac{-1\pm 3}{2}=1$ or -2. If $x=1$ then $y^2=3\Rightarrow y=\pm\sqrt{3}$; if instead $x=-2$ then $y=0$. So the critical pts are

	$(1,\sqrt{3})$	$(1,-\sqrt{3})$	$(-2,0)$
f	$3\sqrt{3}$	$-3\sqrt{3}$	0

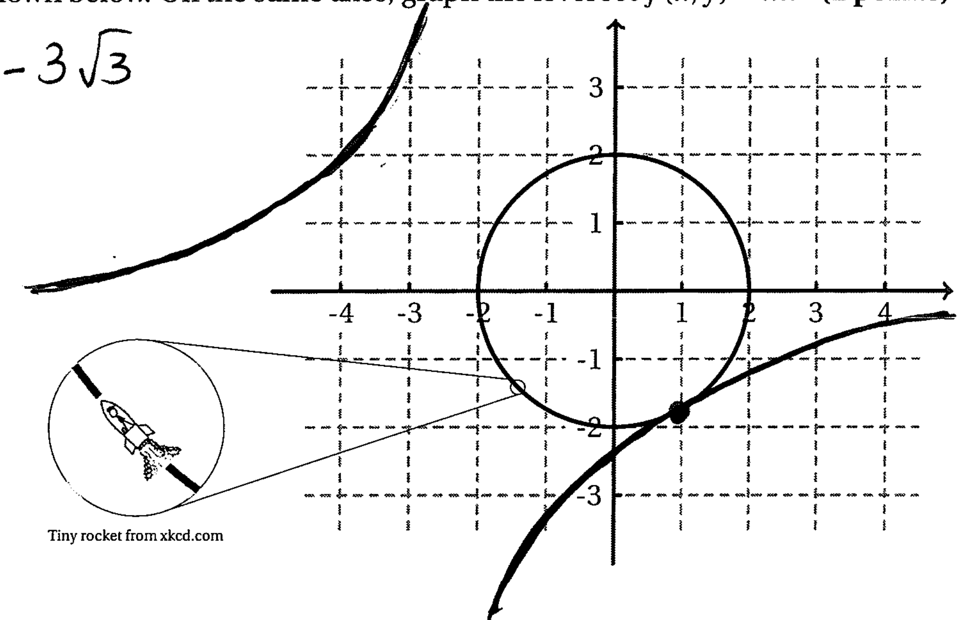
and by the Extreme Value Thm, the absolute min/max must occur amongst these points. So!

$M=3\sqrt{3}$ $m=-3\sqrt{3}$

(b) The orbit of the spaceship is shown below. On the same axes, graph the level set $f(x,y)=m$. **(2 points)**

$$f(x,y)=(x+2)y=-3\sqrt{3}$$

$$\Rightarrow y=\frac{-3\sqrt{3}}{x+2}$$



4. Let C be the curve in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle \sin t,\ 2t,\ \cos t\rangle$ for $0 \le t \le \pi/2$.

(a) Compute the length of C. **(3 points)**

$$\vec{r}\,'(t) = \langle \cos t, 2, -\sin t\rangle \quad |\vec{r}\,'(t)| = \sqrt{\cos^2 t + 4 + \sin^2 t}$$

$$= \sqrt{5}$$

$$L = \int_0^{\pi/2} |\vec{r}\,'(t)|\,dt = \int_0^{\pi/2} \sqrt{5}\,dt$$

$$= \frac{\sqrt{5}}{2}\pi$$

Length = $\sqrt{5}\pi/2$

(b) Evaluate the integral $\int_C x^2 z\, ds$. **(4 points)**

$$ds = |\vec{r}\,'(t)|\,dt$$

$$\int_C x^2 z\, ds = \int_0^{\pi/2} \sin^2 t \cos t\, \overbrace{\sqrt{5}\, dt}$$

$$u = \sin t \quad du = \cos t\, dt$$

$$= \sqrt{5}\int_0^1 u^2\,du = \frac{\sqrt{5}}{3}u^3\Big|_{u=0}^{1} = \frac{\sqrt{5}}{3}.$$

$\int_C x^2 z\, ds$ = $\sqrt{5}/3$

(c) What is the average value of $f(x, y, z) = x^2 z$ on the curve C? **(1 point)**

$$\text{Ave} = \frac{1}{\text{Length}}\int_C f\, ds = \sqrt{5}/3 \Big/ \frac{\sqrt{5}\,\pi}{2}$$

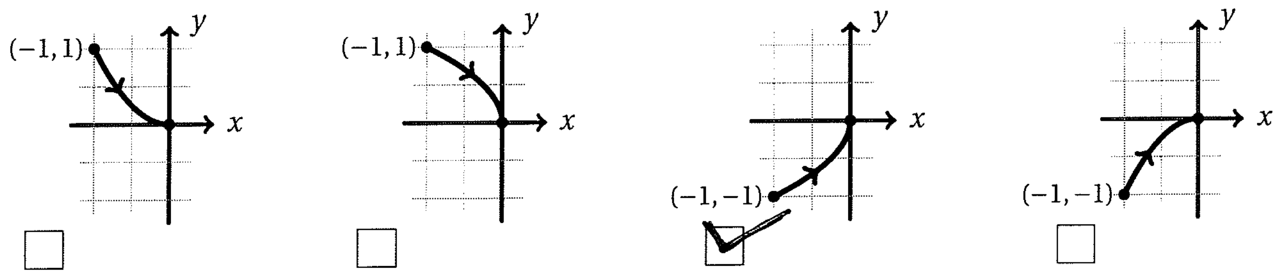
$$= 2/3\pi$$

Average = $2/3\pi$

5. Let C be the curve in $\mathbb{R}^2$ parameterized by $\mathbf{r}(t) = \langle -t^2, t\rangle$ for $-1 \le t \le 0$.

(a) Mark the picture of C from among the choices below. **(1 point)**

The curve describes part of the graph $x=-y^2$ and the y coordinate is non-positive



(b) For the vector field $\mathbf{F} = \langle y,\ x+3\rangle$ directly calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ using the given parameterization. **(4 points)**

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{0} \langle y, x+3\rangle \cdot \vec{r}\,'(t)\,dt$$

$$= \int_{-1}^{0} \langle t, -t^2+3\rangle \cdot \langle -2t, 1\rangle\, dt$$

$$= \int_{-1}^{0} -2t^2 - t^2 + 3\, dt = \int_{-1}^{0} -3t^2+3$$

$$\vec{r}\,'(t) = \langle -2t, 1\rangle \qquad = -t^3+3t\Big|_{t=-1}^{0} = 0-(-(-1)^3-3)$$

$$= 2$$

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = 2}$$

(c) The vector field $\mathbf{F}$ is conservative. Find $f\colon \mathbb{R}^2 \to \mathbb{R}$ with $\nabla f = \mathbf{F}$. **(2 points)**

From $\nabla f = \langle y, x+3\rangle$ get $f = \int y\, dx = xy + C(y)$.

Then $f_y = x + \frac{\partial}{\partial y} C(y) \overset{=}{\text{goal}} x+3 \Rightarrow \frac{\partial C}{\partial y} = 3 \Rightarrow C = 3y$.

So can take $f = xy + 3y$

$$\boxed{f(x,y) = xy + 3y}$$

(d) Use your answer in part (c) to check your answer from part (b). **(2 points)**

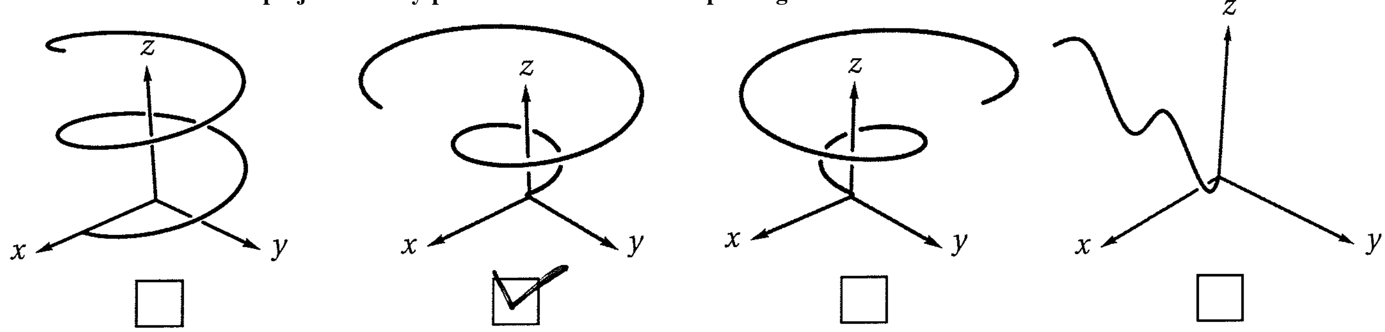
By the Fund. Thm, have

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A) = f(0,0) - f(-1,-1)$$

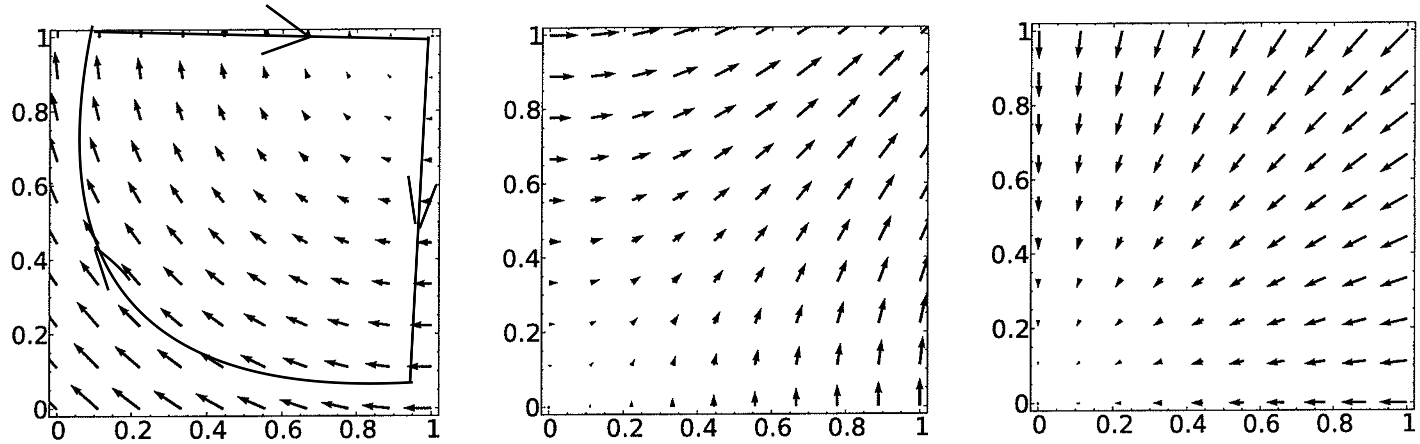
$$= 0 - (1-3) = 2.$$

6. (a) Mark the picture of the curve in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle t\cos t,\ t\sin t,\ t\rangle$ for $0 \le t \le 4\pi$. **(2 points)**

The projection to xy-plane is counterclockwise spiraling outward.



(b) Consider the three vector fields **E**, **F**, and **G** on $\mathbb{R}^2$ shown below.



E **F** **G**

(i) One of these vector fields is $y\mathbf{i} + x\mathbf{j}$. Circle its name here: | E | F | G | **(1 points)**

At (.5,.5), the vector is (.5,.5)

(ii) Exactly one of these vector fields is **not** conservative. Circle it here: | E | F | G | **(1 points)**

Integrating over the closed curve in E is non-zero

(iii) Exactly one of the following is a flowline (also called a streamline or integral curve) for **G** parameterized by time for $0 \le t \le 1$. Circle it. **(1 point)**

$\mathbf{r}(t) = \langle t, 1-t\rangle$	$\mathbf{r}(t) = \langle e^{-t}, e^{-t}\rangle$	$\mathbf{r}(t) = \langle t, \sqrt{t}\rangle$	$\mathbf{r}(t) = \langle 1-t, 1-t\rangle$

As t increases, the path moves toward the origin along the line y = x, and its speed is slowing down

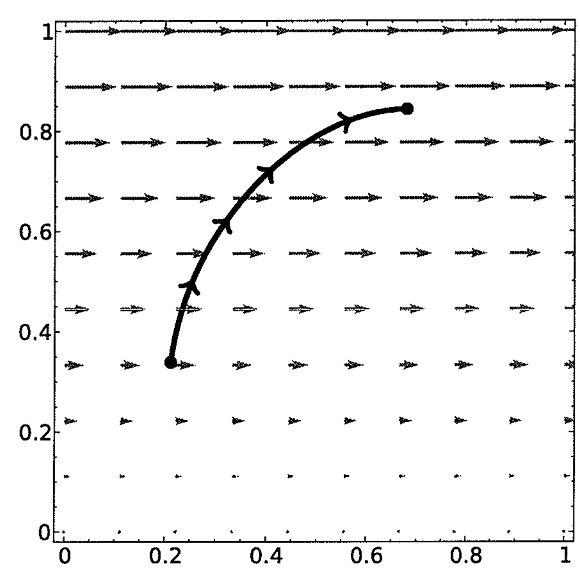
(c) Consider the curve C and vector field **H** shown at right.

Is the integral $\int_C \mathbf{H}\cdot d\mathbf{r}$:

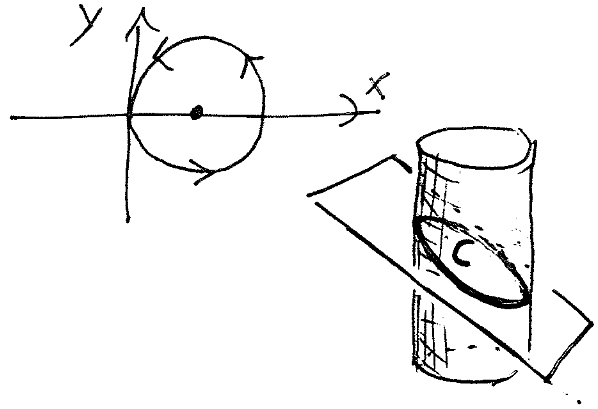
positive	negative	zero

(1 points)

The dot product of the tangent vector and vector field is positive since they make an angle less than pi/2. So the integrand is positive



7. Find a parameterization $\mathbf{r}(t)$ of the curve of intersection of the cylinder $(x-1)^2+y^2=1$ and the plane $x-y+z=1$; specify the range of the parameter t. **(4 points)**



First, we param. the given circle via $x = 1 + \cos t$ and $y = \sin t$.

Then the eqn for the plane gives

$z = 1 - x + y = \sin t - \cos t$

$\boxed{\mathbf{r}(t) = \langle 1+\cos t,\ \sin t,\ \sin t - \cos t \rangle}$ for $\boxed{0} \le t \le \boxed{2\pi}$

8. Consider the function $f(x,y)$ on the rectangle $D = \{0 \le x \le 3 \text{ and } 0 \le y \le 2\}$ whose contours are shown below right. For each part, circle the best answer. **(1 point each)**

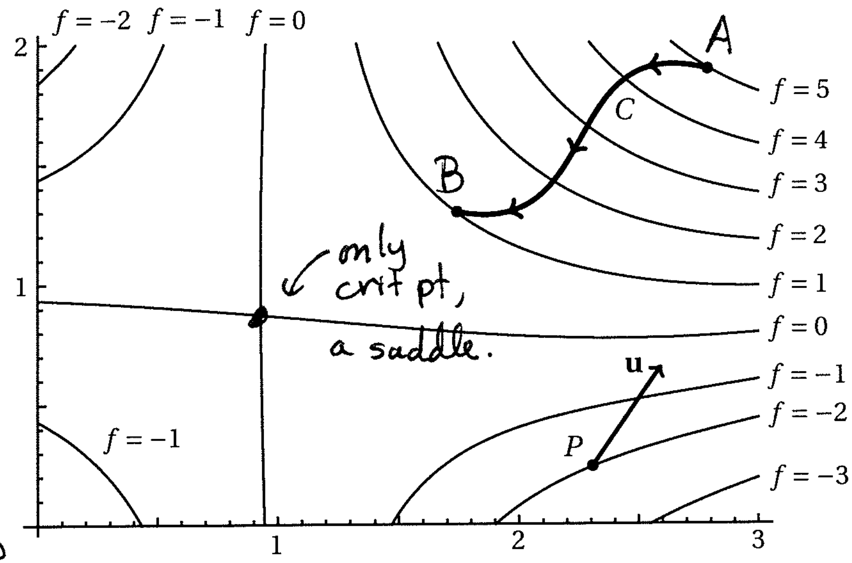
(a) The value of $D_{\mathbf{u}} f(P)$ is:

negative	zero	positive

since f increases in dir $\vec{u}$.

(b) The number of critical points of f in D is:

0	1	2	3



(c) The integral $\int_C f\, ds$ is:

negative	zero	positive

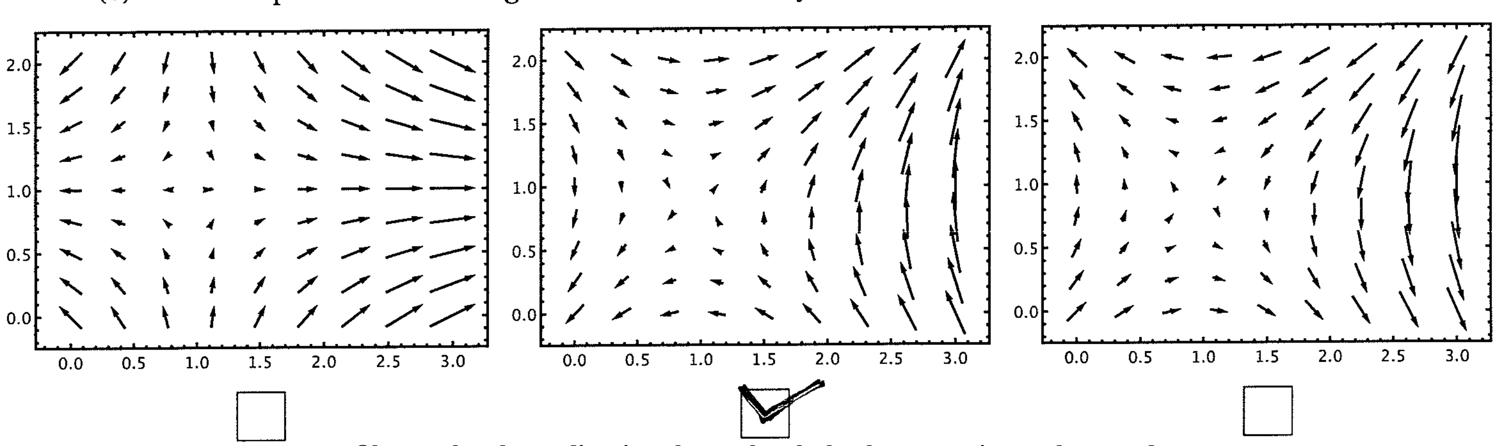
since integrand is always between 1 and 5.

(d) The integral $\int_C \nabla f \cdot d\mathbf{r}$ is:

−4	−2	0	2	4

$\int_C \nabla f \cdot dr = f(B) - f(A) = 1 - 5 = -4$

(e) Mark the plot below of the gradient vector field ∇f.



Observe that the gradient is orthogonal to the level curves so it must be second or third. Also points in direction f is increasing, so its the second.