2016 MIDTERM 2 SOLUTIONS

Jump to problem: (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12).

Problem 1. (back to top)

Answer: 10

Solution: We need to optimize the function $f(x, y) = 3x+y$ with the *constraint equation* $g(x, y) = 10$ where $g(x, y) = x^2 + y^2$. This tells us that we need to use Lagrange multipliers rather than the second derivative test. The first step then is to take gradients:

$$\begin{aligned} \nabla f &= \langle 3, 1\rangle \\ \nabla g &= \langle 2x, 2y\rangle \end{aligned}$$

Taking the coordinates of $\nabla f = \lambda \nabla g$, we get the equations

$$\begin{aligned} 3 &= \lambda 2x \\ 1 &= \lambda 2y \end{aligned}$$

Don't forget that we also have our original constraint:

$$x^2 + y^2 = 10$$

From our Lagrange multiplier equations, we can see that λ, x, and y must be nonzero—if they were, then the left hand sides couldn't be 3 and 1. This means that we can divide by any of these variables without thinking. Solving for λ, we get

$$\begin{aligned} \frac{3}{2x} &= \lambda \\ \frac{1}{2y} &= \lambda \end{aligned}$$

Setting the left hand sides equal to each other gets us

$$\begin{aligned} \frac{3}{2x} &= \frac{1}{2y} \\ 3y &= x \end{aligned}$$

We can use this equation to sub out the x in the constraint equation:

$$\begin{aligned} x^2 + y^2 &= 10 \\ (3y)^2 + y^2 &= 10 \\ 9y^2 + y^2 &= 10 \\ 10y^2 &= 10 \\ y^2 &= 1 \\ y &= \pm 1 \end{aligned}$$

(don't forget the $\pm$!). The critical points of f along our constraint curve are then $(3, 1)$ and $(-3, 1)$. To find the max, we just need to see which of these two points give us the highest value of f. Since $f(3, 1) = 10$ and $f(-3, -1) = -10$, the max is 10.

Problem 2. (back to top)

Answer:

$(-2,0)$ is	a local minimum
$(0,0)$ is	not a critical point
$(2,0)$ is	a saddle point

Solution: First, $(-2,0)$ is a critical point since $g_x(-2,0) = g_y(-2,0) = 0$. The Hessian determinant is

$$\begin{vmatrix} g_{xx}(-2,0) & g_{xy}(-2,0) \\ g_{xy}(-2,0) & g_{yy}(-2,0) \end{vmatrix} = \begin{vmatrix} 12 & 0 \\ 0 & 2 \end{vmatrix} = 12 \cdot 2 - 0 \cdot 0 = 24$$

This only tells us that $(-2, 0$ is either a local min or a local max. To determine which one it is, we can just check concavity in either the x or y direction. Since $g_{xx}(-2,0) = 12 > 0$, it is *concave up* and thus $(-2,0)$ is a local min.

Next, $(0,0)$ isn't even a critical point since $g_x(0,0) = 12 \neq 0$.

Finally, $(2,0)$ is a critical point because $g_x(2,0) = g_y(2,0) = 0$. We take the Hessian determinant:

$$\begin{vmatrix} g_{xx}(2,0) & g_{xy}(2,0) \\ g_{xy}(2,0) & g_{yy}(2,0) \end{vmatrix} = \begin{vmatrix} -12 & 0 \\ 0 & 2 \end{vmatrix} = -12 \cdot 2 - 0 \cdot 0 = -24$$

Since it's negative, $(2,0)$ is a saddle point.

Problem 3. (back to top)

Answer: $f(x,y) = xy^2 + y^3$ (plus any constant)

Solution: We need to pretend that $\mathbf{F} = \langle y^2, 2xy + 3y^2 \rangle$ was gotten by taking the gradient of a function $f(x,y)$. The x-coordinate y^2 would've been f_x, so we can integrate with respect to x to get

$$f(x,y) = \int y^2 dx = xy^2 + C(y)$$

for some function $C(y)$.

We now just need to determine $C(y)$. To do that, note that f_y should be the y-coordinate of $\mathbf{F}$. Therefore,

$$\begin{aligned} 2xy + 3y^2 &= f_y(x,y) \\ &= \frac{\partial}{\partial y}\left(xy^2 + C(y)\right) \\ &= 2xy + C'(y) \end{aligned}$$

Subtracting off the $2xy$ from both sides, we get

$$3y^2 = C'(y)$$

Finally, we can determine $C(y)$ by integrating both sides with respect to y:

$$\begin{aligned} C(y) &= \int 3y^2 dy \\ &= y^3 + C \end{aligned}$$

Altogether, our answer can be $f(x,y) = xy^2 + y^3$ plus any constant.

Problem 4. (back to top)

Answer:

(a) saddle point
(b) $D_{\mathbf{u}}f(P) < D_{\mathbf{u}}f(Q)$
(c) 12
(d) -9

Solution:

(1) At the point A, the level set for $f = 0$ crosses itself. In general, points where a level set crosses itself are saddle points. If you're okay with that, then stop reading.

If not, here's a more detailed analysis. Notice that f decreases when you move up and down from A in the y-direction, so it's a local max in the y-direction and thus $f_{yy}(A) < 0$. However, as you move left to right through A, it's increasing and getting steeper. This tells us that $f_{xx}(A) > 0$. The Hessian determinant at A is

$$f_{xx}(A)f_{yy}(A) - f_{xy}(A)^2$$

We've just showed that $f_{xx}(A)f_{yy}(A) < 0$, and note that $f_{xy}(A)^2 \geq 0$ since it's a square. Altogether then, the Hessian determinant will be negative.

(2) We are now considering the change in f as you travel through P and Q in the direction of $\mathbf{u}$. At both points, f is *decreasing* in the $\mathbf{u}$ direction. The next lower level set for both points in the direction of $\mathbf{u}$ is $f = 0$. Notice as well that both P and Q lie on the level set $f = 4$. In the direction of $\mathbf{u}$, it looks like f drops from 4 to 0 much quicker at P than at Q. This tells us that it's decreasing faster at P, so $D_{\mathbf{u}}f(P)$ is *more negative* than $D_{\mathbf{u}}f(Q)$.

(3) The question is asking for the line integral of a *gradient vector field* ∇f. By the fundamental theorem of calculus for line integrals, this is just f evaluated at the end of C minus f evaluated at the start of C. The ending point is on the level set $f = 0$ while the starting point is on the level set $f = -12$, so the answer is $0 - (-12) = 12$.

(4) The problem now wants you to estimate the line integral of a *function*. The integral is the area of the shower curtain between the graph of f and the curve. This can be done by computing a Riemann sum, which is a fancy way of saying approximating the shower curtain using rectangles. Specifically, we can approximate it by (1) breaking the curve into pieces, (2) taking the length of each piece as the width of our rectangles, and (3) finding a nice value of f over each piece to give us our heights.

   (1) We can break C apart according to its pieces going between adjacent level sets: there is a piece from $f = -12$ to $f = -8$, from $f = -8$ to $f = -4$, and from $f = -4$ to $f = 0$.
   (2) The length the piece from $f = -12$ to $f = -8$ is about $1/3$, from $f = -8$ to $f = -4$ is about $1/2$, and from $f = -4$ to $f = 0$ is about $5/6$.
   (3) Over each piece, we can pick our heights to be the average between the adjacent level sets: between $f = -12$ and $f = -8$ it's -10, between $f = -8$ and $f = -4$ it's -6, and between $f = -4$ and $f = -0$ it's -2.

Putting this all together, we have

$$\begin{aligned} \int_C f d\mathbf{s} &\approx \frac{1}{3} \cdot (-10) + \frac{1}{2} \cdot (-6) + \frac{5}{6} \cdot (-2) \\ &= \frac{-20 - 18 - 10}{6} \\ &= \frac{-48}{6} \\ &= -8 \end{aligned}$$

The closest answer is -9.

Problem 5. (back to top)

Answer: $\langle x^2, y^2 \rangle$

Solution: For a vector field $\mathbf{F} = \langle f(x,y), g(x,y) \rangle$, recall that being conservative means

$$f_y(x,y) = g_x(x,y)$$

I'd check the simpler looking ones first. A pneumonic for this is that gradient vector fields are conservative—the equality above is Clairaut's theorem!

Problem 6. (back to top)

Answer: Only $f(1,2) < 5$ is true.

Solution: We are consider f on the *bounded, closed* region D. By the *extreme value theorem*, f achieves a min and a max over D. Moreover, we know that it'll achieve them either on the boundary circle $\{x^2 + y^2 = 100\}$ or on the critical points of f inside D. From the first two bullet points, we know that f definitely doesn't achieve its minimum on the boundary. Thus, it has to achieve it's minimum on the lone critical point $(1,2)$. Since $f(0,0) = 5$ and $(1,2)$ is the minimum, $f(1,2) < 5$.

Problem 7. (back to top)

Answer: (a) R_3 (b) R_1

Solution: Closed means you include all of your boundary while open means you don't. R_1 and R_4 are open since they exclude their boundary while R_2 is closed since it includes its boundary. R_3 is neither since it contains only some of its boundary.

Simply connected means you don't contain any holes. R_1 has a big gaping hole in it so it's not simply connected.

Problem 8. (back to top)

Answer: The third vector field is not conservative

Solution: The thing to check is the integral of the vector field along a closed path (i.e. a path that starts and ends at the same point) is always zero. One thing to keep in mind is that in the line integral of a vector field, you're integrating the dot product of the vector field with the velocity vector of your parametrized curve—this dot product will be bigger when the vector field aligns with the direction of the curve, negative when they're close to being opposite, and close to zero if they're close to being orthogonal. In the third vector field, consider the integral over some rectangular path going around most of the picture in the clockwise direction. The vector field doesn't change when you move horizontally, so the integral over the right side will cancel out with the integral over the left side since they're opposite directions. The integral over the bottom side will be zero since the vector field is orthogonal to the path. Finally, over the top, the integral will be positive since you're moving left to right (clockwise). Altogether then, the integral will be positive.

Another way to think of this is to check that, letting the vector field be $\langle P, Q\rangle$, whether $P_y = Q_x$. If the vector field is conservative, then this must hold by Clairaut's theorem. The x-coordinates of the vector fields as you travel up are *increasing*, so $P_y(x, y) > 0$. On the other hand, the vectors are the same as you move right, so $Q_x(x, y) = 0$. Therefore, the vector field can't be conservative.

Problem 9. (back to top)

Answer: $\mathbf{r}(t) = \langle 4\sin^2 t - 4\cos^2 t, 2\cos t, 2\sin t\rangle$ for $0 \le t \le 2\pi$

Solution: The starting point is the remember how to parametrize the circle $y^2 + z^2 = 4$ *in the yz-plane*:

$$y = 2\cos t, \quad z = 2\sin t, \quad 0 \le t \le 2\pi$$

Now in 3d, $y^2 + z^2 = 4$ carves out a *cylinder*. What's the intersection with $x = z^2 - y^2$ doing? It's just telling you how to write the new x coordinate in terms of the (y, z)-coordinates we just parametrized. Therefore, if we parametrize the (y, z) coordinates, the equation $x = z^2 - y^2$ tells us how the x coordinate has to move. Putting this together, we have

$$\begin{aligned} \mathbf{r}(t) &= \langle (z(t))^2 - (y(t))^2, y(t), z(t)\rangle \\ &= \langle 4\sin^2 t - 4\cos^2 t, 2\cos t, 2\sin t\rangle \end{aligned}$$

with $0 \le t \le 2\pi$. Note that there are multiple answers to this problem since you could've parametrized the circle in the yz-plane in multiple ways.

Problem 10. (back to top)

Answer:

(a) the second one
(b) $\frac{1}{2}(e^{16\pi} - 1)$

Solution:

(a) One thing to notice is that the z-coordinate of $\mathbf{r}(t)$ is e^t, which grows really fast.

(b) Setting this up, we have $z = e^t$, so $dz = e^t dt$. This gives us

$$\begin{aligned} \int_C z dz &= \int_0^{8\pi} e^t e^t dt \\ &= \int_0^{8\pi} e^{2t} dt \\ &= \frac{1}{2}\left(e^{2t}\right|_0^{8\pi} \\ &= \frac{1}{2}(e^{16\pi} - 1) \end{aligned}$$

Between the second and third lines, I used a u-substitution $u = 2t$.

Problem 11. (back to top)

Answer: $x + y + 4z = -8$

Solution: The problem wants us to find the equation of a plane. For that, we need the normal vector and a point on the plane. To find the normal vector for a tangent plane to a surface, we need to describe the surface using an equation of the form

$$g(x, y, z) = \text{constant}$$

In this case, we could use $g(x, y, z) = x^3 + y^3 + z^3$. We get the normal vector by evaluating ∇g at the point we want the tangent plane to. Chugging along, we have

$$\nabla g = \langle 3x^2, 3y^2, 3z^2 \rangle$$

so

$$\nabla g(1, -1, -2) = \langle 3, 3, 12 \rangle$$

We can directly use this or divide everything by 3 to get

$$\mathbf{n} = \langle 1, 1, 4 \rangle$$

Our point on the plane will just be the point $(1, -1, -2)$, so we have

$$\begin{aligned} 1(x-1) + 1(y+1) + 4(z+2) &= 0 \\ x - 1 + y + 1 + 4z + 8 &= 0 \\ x + y + 4z &= -8 \end{aligned}$$

If we used $\mathbf{n} = 3, 3, 12$, we would've gotten a different looking but equivalent answer.

Problem 12. (back to top)

Answer: $2\sqrt{5}$

Solution: We'll need to take the line integral of $\rho(x, y, z)$ over C.

$$\text{mass} = \int_C \rho d\mathbf{s}$$

First, we'll need to compute $d\mathbf{s}$:

$$\begin{aligned}
\mathbf{r}(t) &= \langle \sin t, 2\sin t, \sqrt{5}\cos t\rangle \\
\mathbf{r}'(t) &= \langle \cos t, 2\cos t, -\sqrt{5}\sin t\rangle \\
|\mathbf{r}'(t)| &= \sqrt{\cos^2 t + 4\cos^2 + 5\sin^2 t} \\
&= \sqrt{5(\cos^2 t + \sin^2 t)} \\
&= \sqrt{5}
\end{aligned}$$

Therefore, $d\mathbf{s} = \sqrt{5}dt$. Our mass integral is

$$\begin{aligned}
\int_C \rho d\mathbf{s} &= \int_C x d\mathbf{s} \\
&= \int_0^\pi (\sin t)(\sqrt{5}dt) \\
&= \sqrt{5}\left(-\cos t\right|_0^\pi \\
&= \sqrt{5}(1 - (-1)) = 2\sqrt{5}
\end{aligned}$$