1. Consider the vector field $\mathbf{F}(x,y) = \langle y + e^x,\ x - \cos y \rangle$. Find a function $f(x,y)$ such that $\mathbf{F} = \nabla f$. **(2 points)**

$f_x = y + e^x \Rightarrow f = \int y + e^x\,dx = xy + e^x + C(y)$

Then $f_y = x + C_y$. Since we want this to be $x - \cos y$,

get $C_y = -\cos y \Rightarrow C = \int -\cos y\,dy = -\sin y$.

So $f = xy + e^x - \sin y$

$f(x,y) = xy + e^x - \sin y$

***By the Extreme Value Theorem, we need to check if the set is closed and bounded — then there MUST be a max/min. Otherwise, there might or might not have one.

2. For each of the given regions D in $\mathbb{R}^2$ below, circle the phrase that makes the sentence true. **(1 point each)**

(a) A continuous function on $D = \{x^2 + 4y^2 \geq 5\}$

must	might or might not	must not

have an absolute maximum. ***This set is not bounded

(b) A continuous function on $D = \{x^2 + 4y^2 \leq 5\}$

must	might or might not	must not

have an absolute maximum.

(c) A continuous function on $D = \{x^2 + 4y^2 < 5\}$

must	might or might not	must not

have an absolute maximum. ***This set is not closed

(d) A continuous function on $D = \{x^2 + 4y^2 = 5\}$

must	might or might not	must not

have an absolute maximum.

3. Suppose $f(x,y)$ is a differentiable function with continuous second order partial derivatives and values given by the table below.

(x,y)	$f(x,y)$	$f_x(x,y)$	$f_y(x,y)$	$f_{xx}(x,y)$	$f_{yy}(x,y)$	$f_{xy}(x,y)$
$(-1,0)$	4	0	0	-2	-3	2
$(0,1)$	0	0	1	1	2	0
$(2,1)$	-2	0	0	1	1	3

$(-1,0)$: $\begin{vmatrix} -2 & 2 \\ 2 & -3 \end{vmatrix} = 2$

$f_{xx} < 0$

$(2,1)$: $\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8 < 0$

For each of the given points, circle the best description of the point. **(1 point each)**

$(-1,0)$	not critical	local minimum	local maximum	saddle point	undetermined
$(0,1)$	not critical	local minimum	local maximum	saddle point	undetermined
$(2,1)$	not critical	local minimum	local maximum	saddle point	undetermined

f_y is not zero

4. Find the maximum and minimum values of the function $f(x, y) = -x + 2y$ on the curve $x^2 + 2y^2 = 3$. **(5 points)**

Set $g(x,y) = x^2 + 2y^2$. We seek the min/max of f subject to the constraint $g = 3$. Use Lagrange Mult: $\nabla f = \langle -1, 2\rangle = \lambda \nabla g = \lambda\langle 2x, 4y\rangle$

$\Rightarrow -1 = 2\lambda x$ and $2 = 4\lambda y$. Note from these eqn's, λ cannot be zero, and so they give

$$\frac{1}{\lambda} = -2x \text{ and } \frac{1}{\lambda} = 2y \Rightarrow -2x = 2y$$

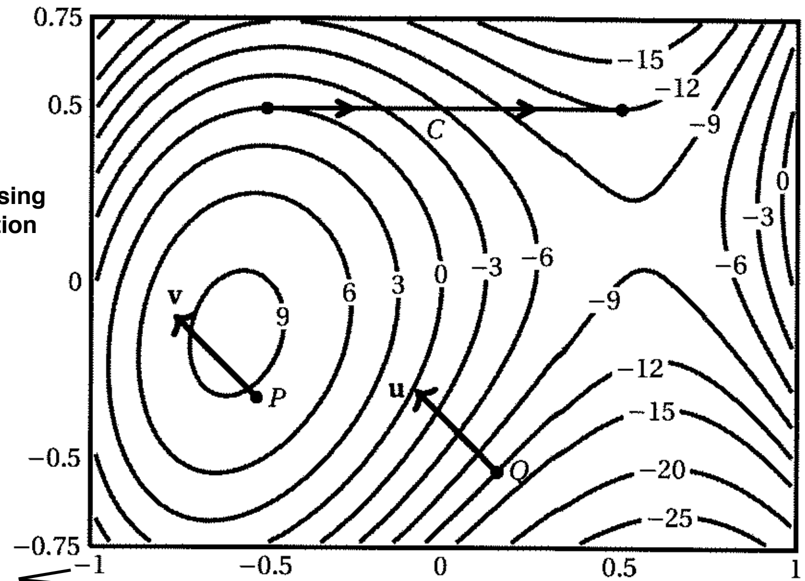
$\Rightarrow y = -x$. Combining w/ our constraint $g = 3$, we get $3 = x^2 + 2(-x)^2 = 3x^2 \Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$. Thus there are two crit pts

$(x,y) = (1,-1)$ where $f = -3$ and $(x,y) = (-1,1)$ where $f = 3$. As the curve $x^2 + 2y^2 = 3$ is an ellipse, it is closed and bounded; as f is cont. The Extreme Value Thm applies to tell us that abs min/max must exist and hence must be amongst the pts identified. Thus

Minimum value =	-3
Maximum value =	$+3$

5. The contour map of a differentiable function g is shown at right. For each part, circle the best answer. **(2 points each)**



(a) The directional derivative $D_{\mathbf{v}}g(P)$ is:

positive	negative	zero

***g is increasing in this direction

(b) The vector $\mathbf{u}$ is parallel to $\nabla g(Q)$.

True	False

***u is perpendicular to the level sets, as is the gradient

(c) Estimate $\int_C g(x,y)\,ds$.

−12	−9	−6	−3	0	3	6	9	12

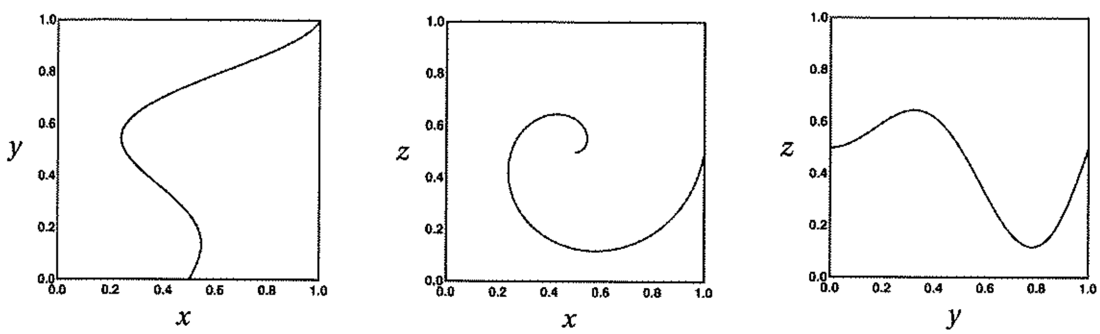
**g is decreasing at a roughly constant rate from zero to -12, so the average is approximatley -6. The length is one, so the integral is -6.

(d) Find $\int_C \nabla g \cdot d\mathbf{r}$:

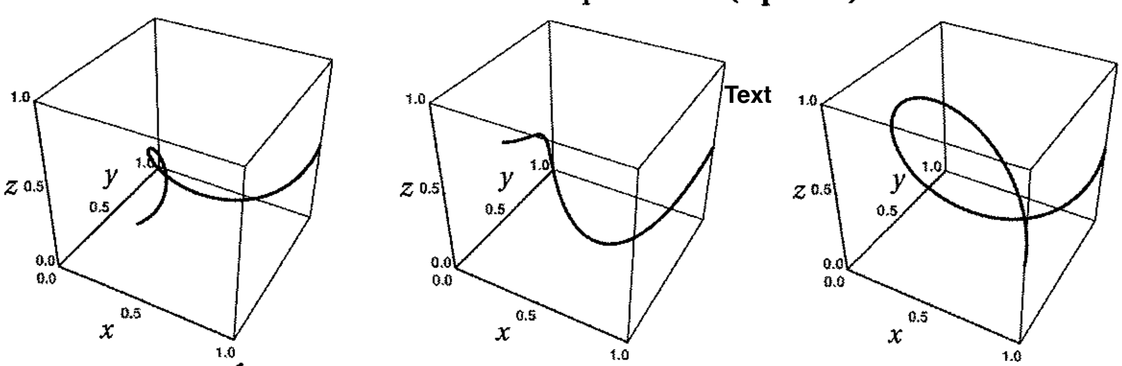
−12	−9	−6	−3	0	3	6	9	12

***This is the fundamental Theorem of Line Integrals

6. Consider the curve C in $\mathbb{R}^3$ whose projections onto the xy, xz and yz planes are:



Check the box below the three-dimensional plot of C. **(2 points)**



**For this curve, the xz-projection does not end at (.5,.5), as the one above does.

***For this curve, the xz-projection should start and end at the same points (it should look like a closed curve).

8. (a) Let S_1 be the surface defined by $x = 2z^2 + y^2$ (an elliptic paraboloid). Find an equation for the tangent plane to S_1 at the point $(3, -1, 1)$. **(4 points)**

Set $f = -x + y^2 + 2z^2$ so $S_1 = \{f = 0\}$. Normal to tangent plane is $\nabla f = \langle -1, 2y, 4z \rangle$ at $(3,-1,1)$, i.e. $\vec{n} = \langle -1, -2, 4 \rangle$. Equation is thus

$$-1(x-3) + (-2)(y+1) + 4(z-1) = 0 \iff$$

$$-x - 2y + 4z = 3$$

Equation: $\boxed{-1}\,x + \boxed{-2}\,y + \boxed{4}\,z = \boxed{3}$

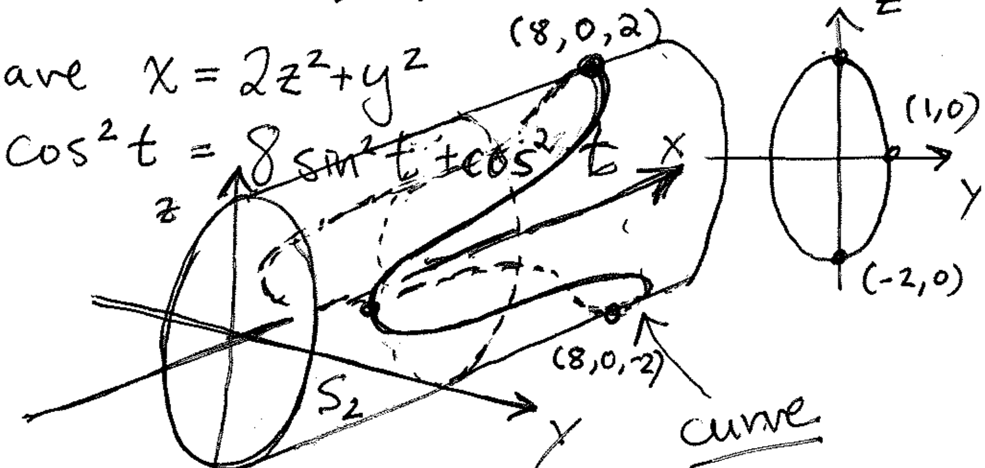
(b) Let S_1 be as in the previous part, and let S_2 be the surface defined by $y^2 + \frac{z^2}{4} = 1$ (a cylinder over an ellipse). Find a vector function $\mathbf{r}(t)$ that parametrizes the curve that is the intersection of the surfaces S_1 and S_2. Specify the range of the parameter values so that the function traces the curve exactly once. **(4 points)**

Param. the ellipse $y^2 + \frac{z^2}{4} = 1$ in the yz-plane by $y = \cos t$ and $z = 2\sin t$ for $0 \le t \le 2\pi$.

Now on S_1 have $x = 2z^2 + y^2$

$= 2 \cdot (2\sin t)^2 + \cos^2 t = 8\sin^2 t + \cos^2 t$

$= 7\sin^2 t + 1$



$\mathbf{r}(t) = \langle 7\sin^2 t + 1,\ \cos t,\ 2\sin t \rangle$ for $\boxed{0} \le t \le \boxed{2\pi}$

7. Let C be the curve in three-dimensional space parametrized by $\mathbf{r}(t) = \langle 2\cos t,\ 2\sin t,\ t\rangle$ for $-\pi \le t \le \pi$.

(a) Find the mass of a thin wire in the shape of C, if the density function is $\rho(x, y, z) = x + z + 10$. **(5 points)**

$$\text{Mass} = \int_C x+z+10 \; ds = \int_{-\pi}^{\pi} (2\cos t + t + 10)\,|\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,'(t) = (-2\sin t,\ 2\cos t,\ 1)$$

$$|\vec{r}\,'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$$

$$= \sqrt{5}\left(2\sin t + \frac{t^2}{2} + 10t\right)\Bigg|_{t=-\pi}^{\pi}$$

$$= 20\sqrt{5}\,\pi$$

Mass = $20\sqrt{5}\,\pi$

(b) Suppose that a particle moves along C. Find the work done on the particle by the force $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j}$. **(5 points)**

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_{-\pi}^{\pi} \underbrace{(2\sin t, -2\cos t, 0)}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{(-2\sin t, 2\cos t}_{\vec{r}\,'(t)}, 1)\, dt$$

$$= \int_{-\pi}^{\pi} -4\sin^2 t - 4\cos^2 t + 0 \; dt = -4\int_{-\pi}^{\pi} dt$$

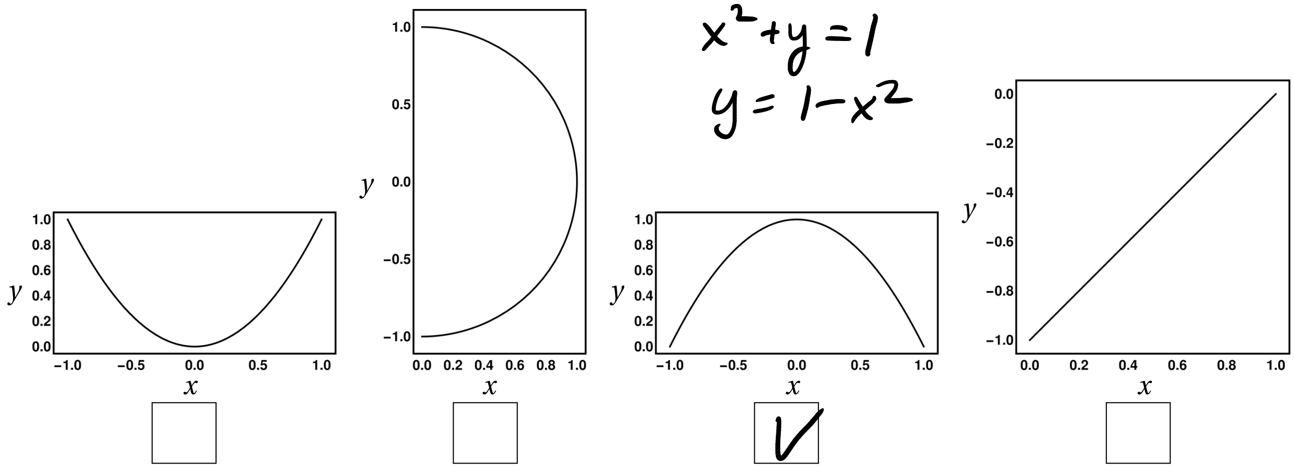
$$= -8\pi$$

Work = -8π

9. Check the box below the picture of the curve $\mathbf{r}(t) = \langle \sin t,\ \cos^2 t \rangle$, $0 \le t \le 2\pi$. **(2 points)**

$x^2 + y = 1$

$y = 1 - x^2$



10. A vector field $\mathbf{G}$ is plotted at right.

(a) Circle the formula for $\mathbf{G}$. **(1 point)**

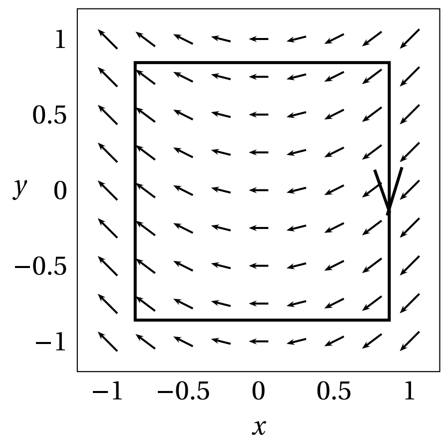
$x\mathbf{i} + y\mathbf{j}$ $-x\mathbf{i} - \mathbf{j}$ $-\mathbf{i} - x\mathbf{j}$ $y\mathbf{i} - \mathbf{j}$

*****Check at (0,1)**

(b) $\mathbf{G}$ is conservative. **(1 point)**

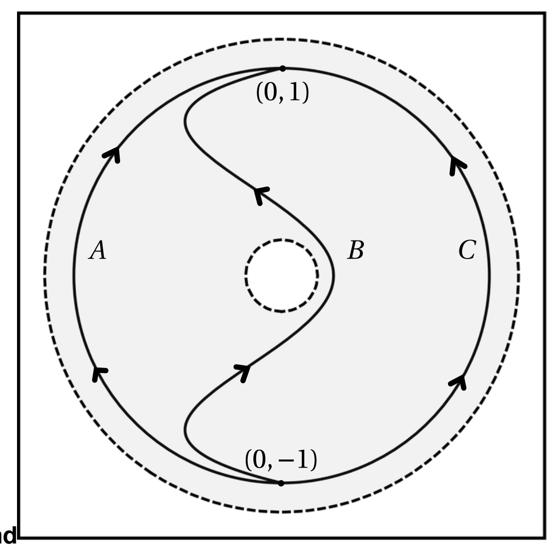
True   False

*****The integral around the closed curve shown (the square) is positive, since the top and bottom cancel, and the sides both contribute positively.**



11. The region D defined by $\{0.03 < x^2 + y^2 < 1.3\}$ is shown at right. Within this region are three curves A, B, C. Each curve starts at $(0,-1)$ and ends at $(0,1)$. Suppose that $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a differentiable vector field defined on D with the properties

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \qquad \int_A \mathbf{F}\cdot d\mathbf{r} = -1, \quad \text{and} \quad \int_C \mathbf{F}\cdot d\mathbf{r} = 2.$$



(a) The region D is simply connected. **(1 point)**

True   False

*****There is a hole in the middle so it cannot be simply connected.**

(b) $\mathbf{F}$ is conservative. **(1 point)**

Yes   No   Cannot determine

*****The integral around the closed curve A.(-C) is -1-2 = -3, which is not 0.**

(c) Find $\int_B \mathbf{F}\cdot d\mathbf{r}$. **(1 point)**

-3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5 3

*****The curves B and C are contained in a simply connected open subset. So on this subset the vector field is conservative, and the integral over B and C are thus equal.**