1. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x - y$. Let C be the circle $x^2 + y^2 = 2$.

(a) Use Lagrange multipliers to find the absolute max and min of f on C. **(5 points)**

Since C is closed and bounded, we know abs min/max exist. To find them, set $g = x^2 + y^2$ so $C = \{g = 2\}$, and consider $\nabla f = \langle 1, -1 \rangle = \lambda \nabla g = \lambda \langle 2x, 2y \rangle$

$$\Rightarrow \begin{array}{l} 1 = \lambda 2x \\ -1 = \lambda 2y \end{array}$$

Now $\lambda \neq 0$ since $\lambda 2x \neq 0$,

and so we can say $\frac{1}{2\lambda} = x$ and $-\frac{1}{2\lambda} = y \Rightarrow y = -x$.

From $g = 2$, we get $x^2 + (-x)^2 = 2 \Rightarrow x^2 = 2 \Rightarrow x = \pm 1$.

So the min/max occur at $(1, -1)$ and $(-1, 1)$

where $f = \quad 2 \quad\quad -2$

Absolute max of f on C is	2	which occurs at the point(s)	$(1, -1)$
Absolute min of f on C is	-2	which occurs at the point(s)	$(-1, 1)$

(b) Find the absolute max of f on $\mathbb{R}^2$ if it exists. **(1 point)**

As $f(x, 0) = x$ is unbounded as $x \to \infty$ we see there is no abs. max.

Absolute max of f on $\mathbb{R}^2$ is DNE

2. Consider the differentiable function $f(x, y)$ on the rectangle $D = \{0 \le x \le 6 \text{ and } 0 \le y \le 4\}$ whose contours are shown below right. For each part, circle the best answer. **(1 point each)**

(a) The maximum value of f on D is:

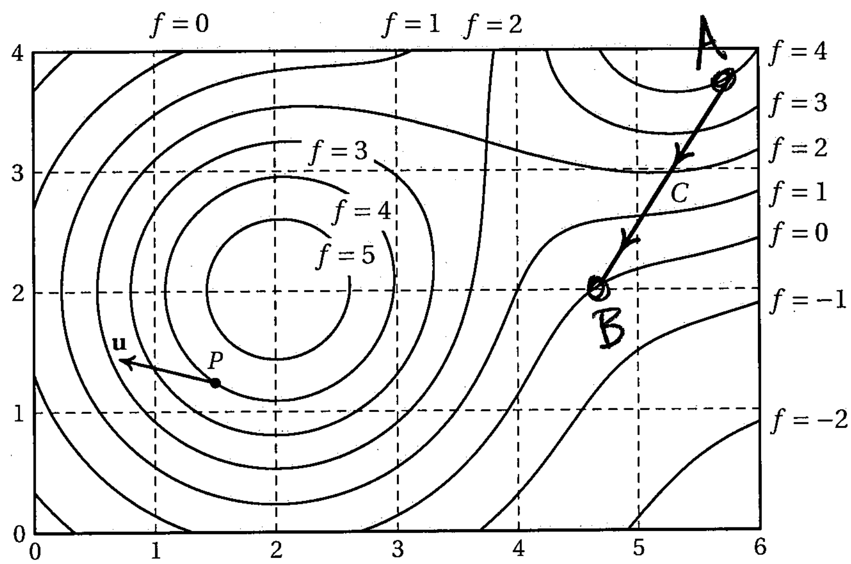
0	3	6	9	DNE

(b) The value of $D_{\mathbf{u}} f(P)$ is:

negative	zero	positive

(c) The number of critical points of f in D which are saddles is:

0	1	2	3



(e) The integral $\int_C f \, ds$ is:

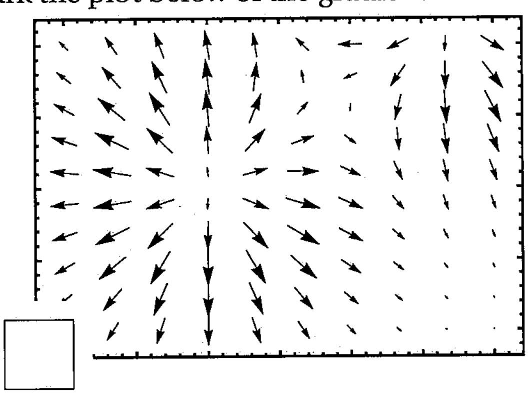
−8	−4	−2	0	2	4	8

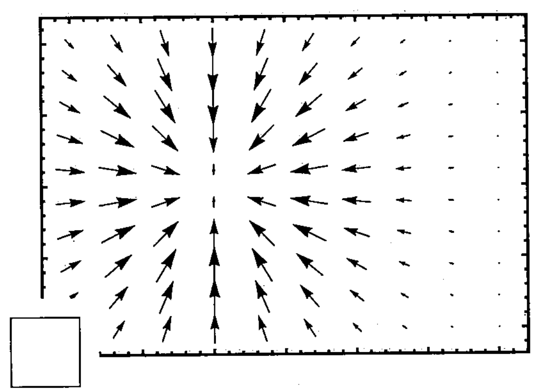
(f) The integral $\int_C \nabla f \cdot d\mathbf{r}$ is:

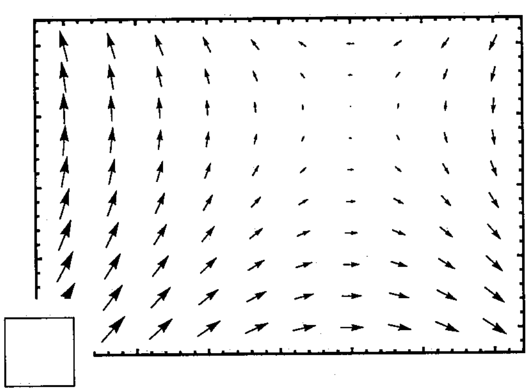
−6	−4	−2	0	2	4	6

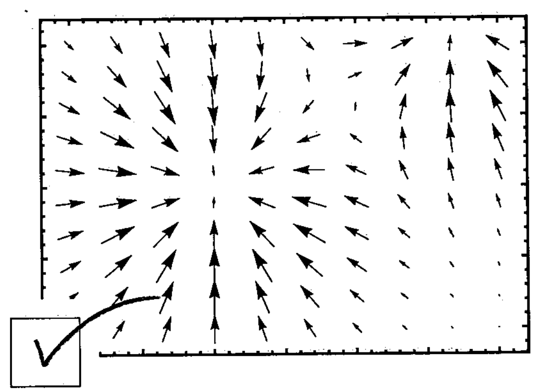
$= f(B) - f(A)$
$= 0 - 4 = -4$

(h) Mark the plot below of the gradient vector field ∇f.







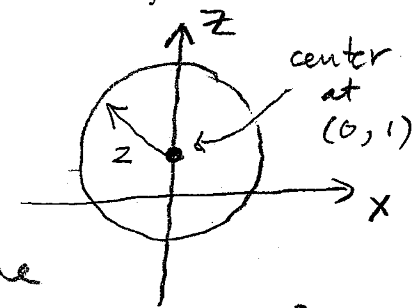


✓

(e): $\int_C f \, ds = \text{len}(C) \cdot$ Average of f on C $\approx 2 \cdot 2 = 4$

3. Let C be the curve in $\mathbb{R}^3$ that is the intersection between the circular cylinder $x^2 + (z-1)^2 = 4$ and the plane $x + y + z = 1$.

(a) Find a vector function $\mathbf{r}(t)$ that parameterizes C, traversing the whole curve exactly once. Be sure to specify the domain of your parameterization. **(4 points)**



Looking down the y-axis, we see and we can param. that circle by $x = 2\cos t$, $z = 2\sin t + 1$. From the plane eqn we get $y = 1 - x - z = -2(\cos t + \sin t)$

So our answer is

$\mathbf{r}(t) = \langle 2\cos t, -2(\cos t + \sin t), 2\sin t + 1 \rangle$ for $0 \le t \le 2\pi$

(b) The vector $\mathbf{v} = \langle 1, 2, 3 \rangle$ is tangent to C at some point: true / false **(1 point)**

Scratch Space

As C is in the plane, its tangents will always be orthogonal to the normal for the plane, $\vec{n} = \langle 1, 1, 1 \rangle$. But $\vec{n} \cdot \vec{v} = 6 \neq 0$.

4. The vector field $\mathbf{F}(x,y) = \langle y^2+1,\ 2xy+1 \rangle$ on $\mathbb{R}^2$ is conservative. Find a function $f(x,y)$ where $\mathbf{F} = \nabla f$. **(3 points)**

First, $f = \int y^2+1\,dx = xy^2 + x + C(y)$. Then $\frac{\partial f}{\partial y} =$

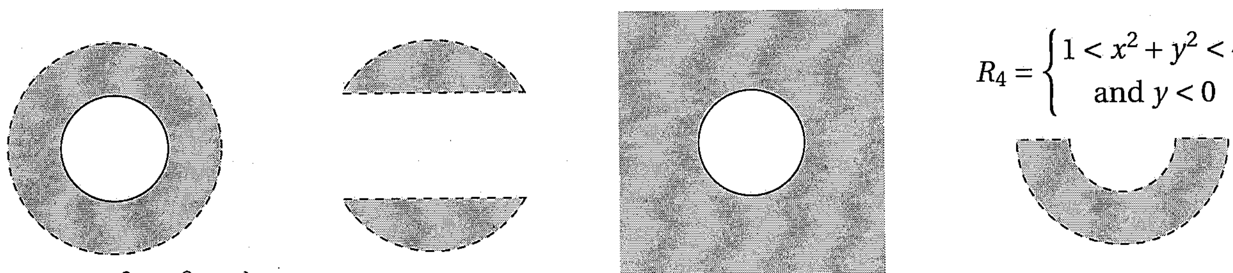
$2xy + \frac{\partial C}{\partial y} \Rightarrow \frac{\partial C}{\partial y} = 1 \Rightarrow C = \int 1\,dy = y$ (+ constant)

So $f = xy^2 + x + y$ works.

$f(x,y) = \boxed{xy^2 + x + y}$

5. Consider the following four regions in the plane: **(1 point each)**

$R_4 = \left\{ \begin{array}{c} 1 < x^2+y^2 < 4 \\ \text{and } y < 0 \end{array} \right\}$



$R_1 = \{1 \le x^2+y^2 < 4\}$ $\quad R_2 = \left\{ \begin{array}{c} x^2+y^2 < 4 \\ \text{and } |y| > 1 \end{array} \right\}$ $\quad R_3 = \{4 \le x^2+y^2\}$

(a) Which region is closed? $\boxed{R_1 \quad R_2 \quad R_3 \quad R_4}$

(b) Which region is simply connected (and thus also connected)? $\boxed{R_1 \quad R_2 \quad R_3 \quad R_4}$

Scratch Space

6. Find the mass of a thin wire in the shape of the curve parameterized by $\mathbf{r}(t) = \langle \sin t,\ 2t,\ \cos t\rangle$ for $0 \le t \le \pi$, if the wire has density function $\rho(x, y, z) = y$. **(4 points)**

$$\text{Mass} = \int_C \rho \, ds = \int_0^{\pi} \rho(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{\pi} 2t\sqrt{5}\, dt = \sqrt{5}\, t^2 \Big|_0^{\pi} = \sqrt{5}\,\pi^2$$

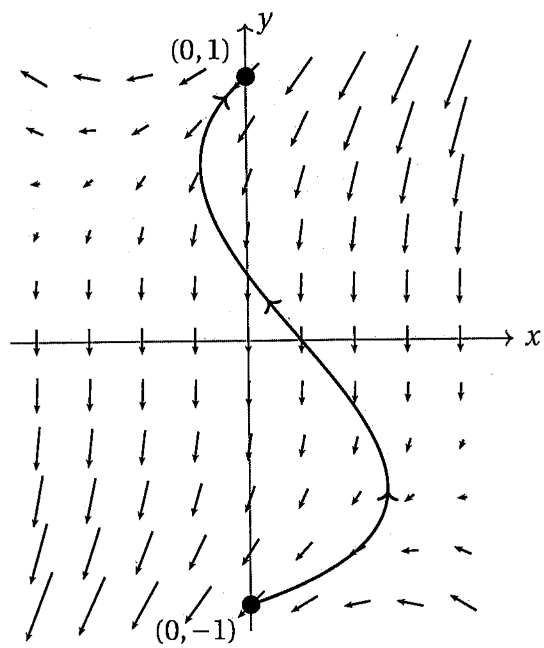
$$\vec{r}\,'(t) = \langle \cos t, 2, -\sin t\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{\cos^2 t + 2^2 + \sin^2 t}$$

$$= \sqrt{5}$$

Mass = $\sqrt{5}\,\pi^2$

7. A vector field **F** is shown at right; for scale, $\mathbf{F}(0,0) = \langle 0, -0.1\rangle$. Assuming that **F** is conservative, circle the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve shown from $(0,-1)$ to $(0,1)$.



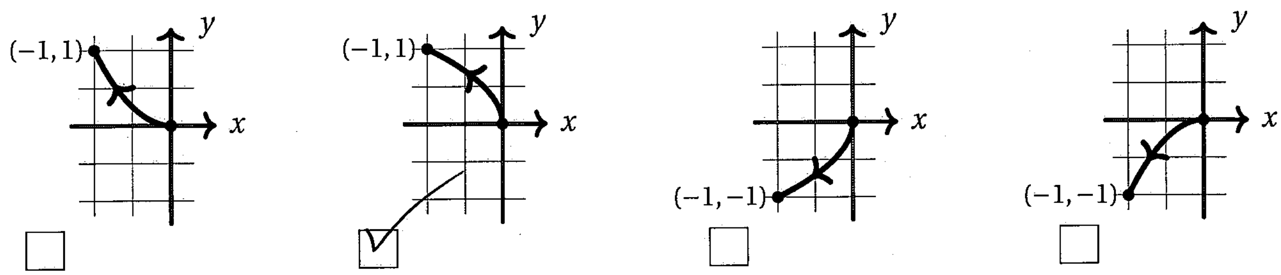
−0.3	−0.2	−0.1	0	0.1	0.2	0.3

(2 points) As f is conserv, this is the same as $\int_L \vec{F}\cdot d\vec{r}$ for L = line from (0, -1) to (0, 1). Along L, vertical comp of $\vec{F}$ appears const, so −0.1, so $\int_L \vec{F}\cdot d\vec{r} = \int_L \vec{F}\cdot\vec{T}\, d\vec{r}$

$$= \int_L \vec{F}\cdot\langle 0,1\rangle\, ds = \int_L -0.1\, ds = -\frac{1}{10}\,\text{len}(L) = -0.2.$$

8. Let C be the curve in $\mathbb{R}^2$ parameterized by $\mathbf{r}(t) = \langle -t^2,\ t \rangle$ for $0 \le t \le 1$.

(a) Mark the picture of C from among the choices below. **(1 point)**



(b) For the vector field $\mathbf{F} = \langle 1,\ x \rangle$, calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(4 points)**

Have $\vec{r}\,'(t) = \langle -2t, 1 \rangle$ so

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle 1, -t^2 \rangle \cdot \langle -2t, 1 \rangle\, dt$$

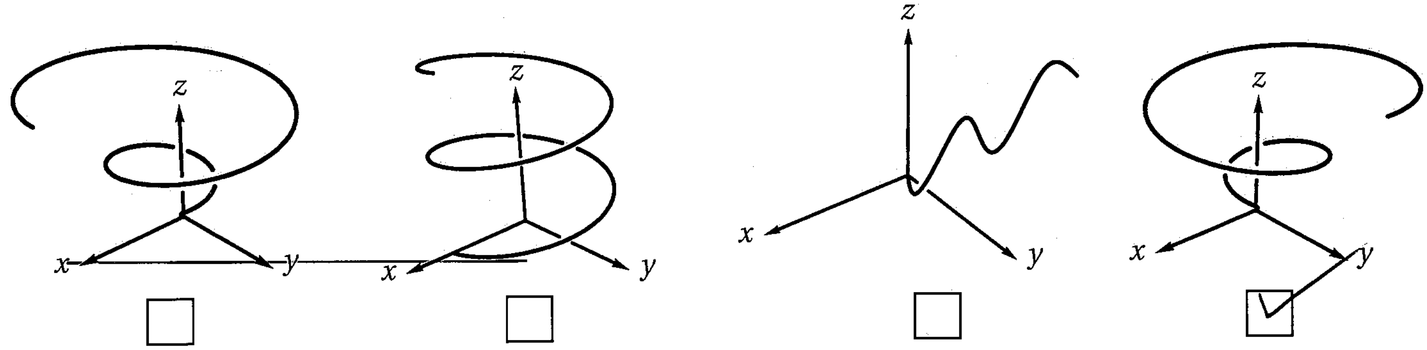
$$= \int_0^1 -2t - t^2\, dt$$

$$= -\left(t^2 + t^3/3\right)\Big|_{t=0}^{t=1} = -4/3$$

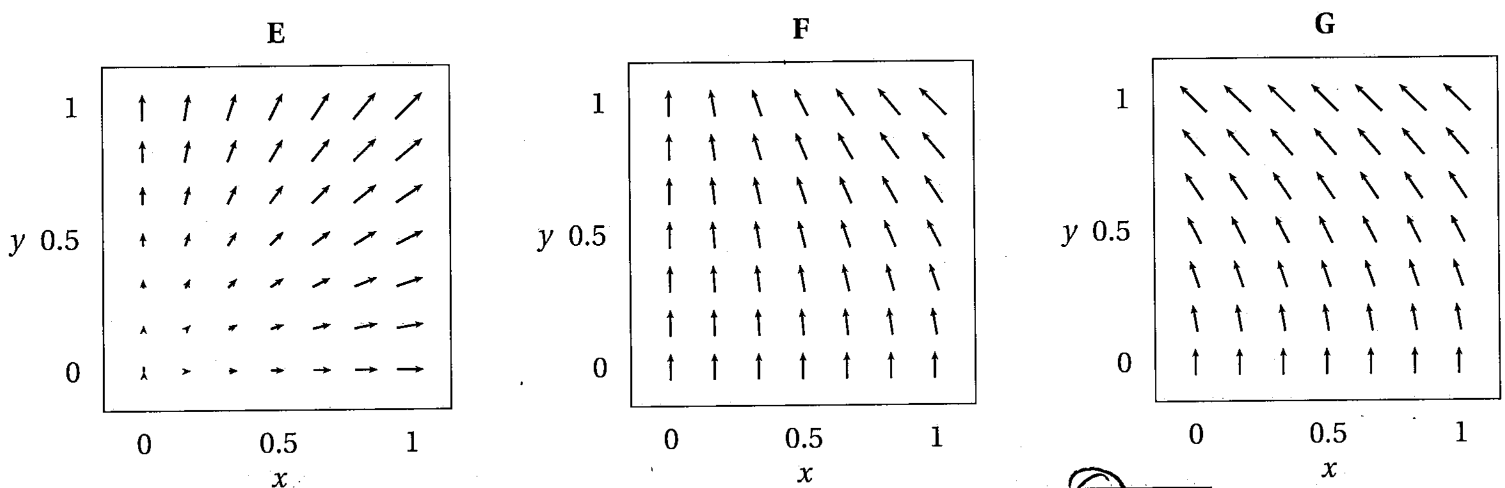
$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = -4/3}$$

$r(4\pi) = \langle 0, 4\pi, 4\pi \rangle$ in yz-plane

9. Mark the picture of the curve in $\mathbb{R}^3$ parameterized by $\mathbf{r}(t) = \langle t\sin t,\ t\cos t,\ t \rangle$ for $0 \le t \le 4\pi$. **(2 points)**



10. Consider the three vector fields **E**, **F**, and **G** on $\mathbb{R}^2$ shown below. **(1 point each)**



(a) One of these vector fields is $\langle -xy, 1 \rangle$. Circle its name here: | **E** | **F** | **G** | **(1 point)**

(b) Exactly one of these vector fields is conservative. Circle it here: | **E** | **F** | **G** | **(1 point)**

(c) Exactly one of the following is a flowline (also called a streamline or integral curve) for **E** parameterized by time for $0 \le t \le 1$. Circle it. **(1 point)**

Either of these answers was accepted as correct as the range of r on the left is not in the picture. It is however the actual flow line for -(infinity) < t < 0.

$\mathbf{r}(t) = \langle t, 1-t \rangle$	$\mathbf{r}(t) = \langle t, \sqrt{t} \rangle$	$\mathbf{r}(t) = \langle t, t \rangle$	$\mathbf{r}(t) = \langle e^t, e^t \rangle$

Scratch Space

don't follow flow lines.

$\vec{r}' = \langle 1, 1 \rangle$

so const. speed, incompatible with $\vec{E}$.