Name: NetID: UID: /53

Section: ED1: Nathan Dunfield (8am) ED3: Ping Hu (9am) ED5: Ping Hu (10am)
ED2: Boonrod Yuttanan (8am) ED4: Jeff Mudrock (9am) ED6: Boonrod Y. (10am)

Instructions: Take care to note that problems are not weighted equally. Calculators, books, notes, and suchlike aids to gracious living are not permitted. **Show all your work** as credit will not be given for correct answers without proper justification, except for the "circle your answer" questions.

Important note: Several of the problems ask you to "completely setup but not evaluate" a certain integral. This means that all the limits of integration are specified, and the integrand is in terms of the final variables. For example, if S is a surface in $\mathbb{R}^3$, then an acceptable answer for setting up $\iint_S (x+y)^2\, dA$ would be something like $\int_0^2 \int_0^{1-v} (u^2 + v \sin u)\, du\, dv$.

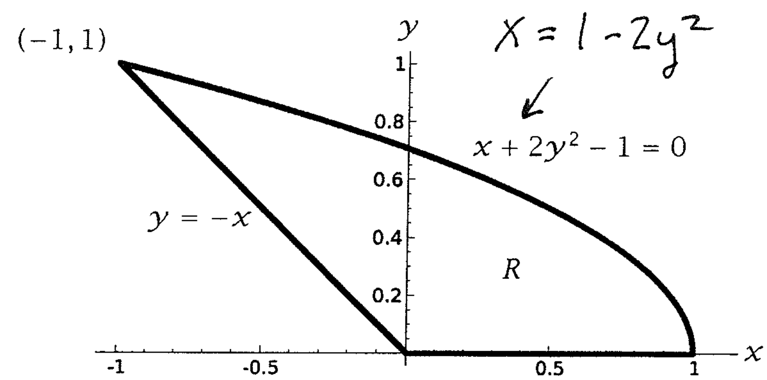
Scratch Space: Below, and on back of last sheet.

Good luck!

Problem	Score	Out of
1		7
2		6
3		10
4		6
5		6
6		11
7		7
Total		53

Solution Key.

1. Let R be the region shown at right. Evaluate $\iint_R y\, dA$. **(7 points)**



$$\iint_R y\, dA$$

$$= \int_0^1 \int_{-y}^{1-2y^2} y\, dx\, dy = \int_0^1 yx \Big|_{x=-y}^{1-2y^2} dx$$

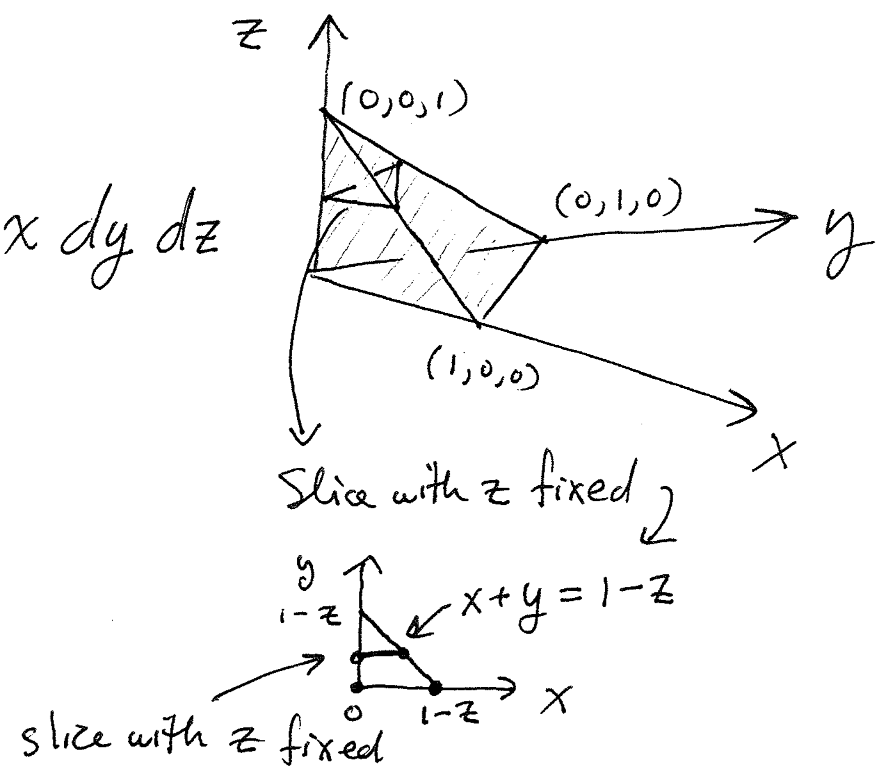
$$= \int_0^1 y(1-2y^2) - (y(-y))\, dy$$

$$= \int_0^1 y^2 - 2y^3 + y\, dy = \frac{y^3}{3} - \frac{2}{4}y^4 + \frac{y^2}{2} \Big|_{y=0}^{1}$$

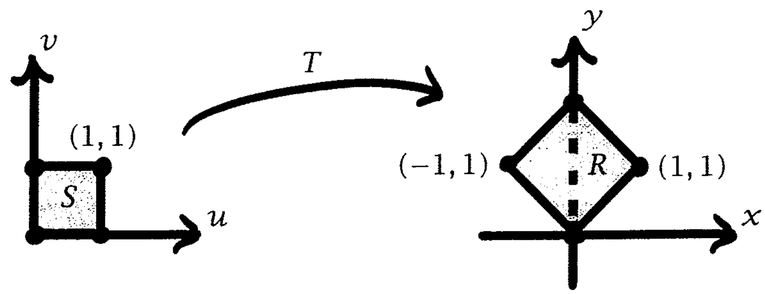
$$= 1/3.$$

2. Consider the solid region E in the positive octant cut off by $x + y + z = 1$. Completely setup, but do not evaluate, a triple integral which computes the volume of E. **(6 points)**

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} 1\, dx\, dy\, dz$$



3. Let R be the region shown at right.



(a) Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [0,1] \times [0,1]$ to R. **(4 points)**

Using a linear transform $T(u,v) = (au+bv,\ cu+dv)$

$$T(1,0) = (a,c) = (1,1) \Rightarrow a = c = 1.$$

$$T(0,1) = (b,d) = (-1,1) \Rightarrow b = -1,\ d = 1$$

<u>So</u>: $T(u,v) = (\underbrace{u-v}_{x},\ \underbrace{u+v}_{y}).$

(b) Use your change of coordinates to evaluate $\iint_R (x+y)^2\, dA$ via an integral over S. **(6 points)**
Emergency backup transformation: If you can't do (a), pretend you got the answer $T(u,v) = (uv, v)$ and do part (b) anyway.

$$\iint_R (x+y)^2\, dA = \iint_S \left((u-v)+(u+v)\right)^2 |\det J|\, du\, dv$$

$$= \int_0^1\int_0^1 4u^2\,(2)\, du\, dv = 8\int_0^1\int_0^1 u^2\, du\, dv$$

$$= 8\int_0^1 \frac{u^3}{3}\Big|_{u=0}^{1} dv$$

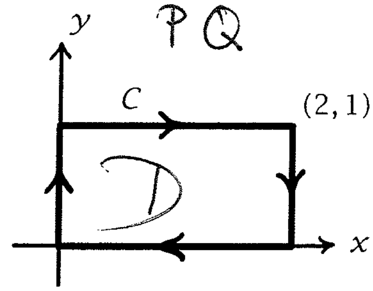
$$= \frac{8}{3}\int_0^1 dv = \boxed{\frac{8}{3}}.$$

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\det = 2.$

4. Let C be the oriented curve in $\mathbb{R}^2$ shown at right. For the vector field $F(x, y) = (x^3, x^2)$, use Green's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(6 points)**



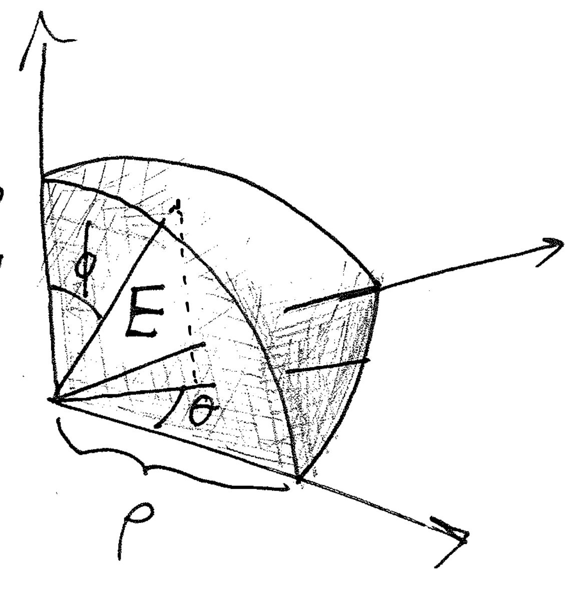
$$\int_C F\cdot d\vec{r} = -\int_{\underbrace{-C}_{\text{C reversed so it's anti-clockwise}}} \vec{F}\cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= -\int_0^1\int_0^2 2x \, dx\, dy = -\int_0^1 x^2\Big|_{x=0}^{2} dy$$

$$= -\int_0^1 4\, dy = \boxed{-4}$$

5. Let E be the portion of the positive octant which is inside the unit sphere. Use spherical coordinates to completely setup, but not evaluate, the integral $\iiint_E x + z\, dV$. **(6 points)**

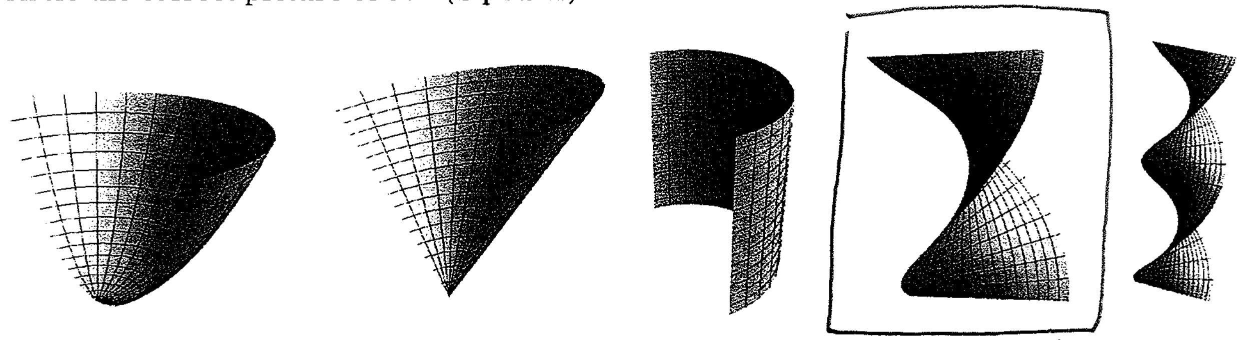
$$\int_0^1\int_0^{\pi/2}\int_0^{\pi/2} (\qquad) \underbrace{\rho^2 \sin\phi \, d\theta\, d\phi\, d\rho}_{dV}$$



$$\rho\sin\phi\cos\theta + \rho\cos\phi$$

6. Let S be the surface in $\mathbb{R}^3$ parameterized by $\mathbf{r}(u,v) = (v\cos u, v\sin u, u)$ for $0 \le u \le \pi$ and $-1 \le v \le 1$.

(a) Circle the correct picture of S. **(2 points)**

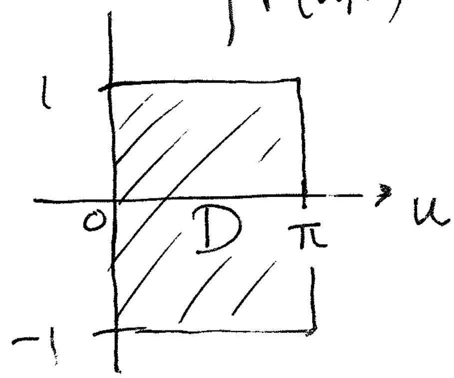


(b) Completely setup, but do not evaluate, the integral $\iint_S y\, dA$. **(6 points)**

$\vec{r}(u,v)$

$$\iint_S y\, dA = \iint_D v\sin u\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \int_0^{\pi}\int_{-1}^{1} v\sin u\sqrt{1+v^2}\, dv\, du$$



$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 1 \\ \cos u & \sin u & 0 \end{vmatrix} = (-\sin u, \cos u, -v)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 u + \cos^2 u + v^2} = \sqrt{1+v^2}$$

(c) Find the equation for the tangent plane to S at the point $(0,0,\pi/2)$ in $\mathbb{R}^3$. **(3 points)**

If $\vec{r}(u,v) = (0,0,\pi/2)$, this tells us that $u = \pi/2$.
As $\vec{r}(\pi/2, v) = (0, v, \pi/2)$, this implies that $v = 0$.
Normal vector to plane is $\vec{r}_u \times \vec{r}_v = (-1, 0, 0)$
at $(u,v) = (\pi/2, 0)$. Hence the plane is $-1(x-0) = 0$,
i.e. the yz-plane.

For part (a): $r(u,v) = (v\cos u,\ v\sin u,\ u)$, $0 \le u \le \pi$, $-1 \le v \le 1$

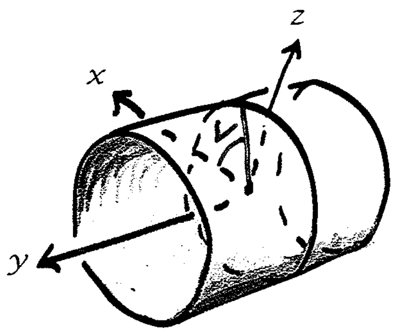
$$\begin{array}{ccc} v\cos u & v\sin u & u \\ \Uparrow & \Uparrow & \uparrow \\ x(u,v) & y(u,v) & z(u,v) \end{array}.$$

$$x^2 + y^2 = (v\cos u)^2 + (v\sin u)^2 = v^2 \qquad \begin{array}{l} 0 \le v^2 \le 1 \\ 0 \le u \le \pi \end{array}$$

thus the projection to x-y plane give you a half disk, since $u \in [0, \pi]$. since $z = u$, which means the graph will be twisted, but it will be just half period. only possible choice is #4.

7. For each surface S below, give a parameterization $\mathbf{r}\colon D \to S$. Be sure to explicitly specify the domain D.

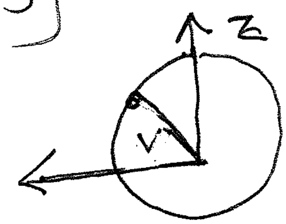
(a) The portion of the cylinder $x^2 + z^2 = 4$ where $-3 \le y \le 3$. **(4 points)**



Parameters $y = u$

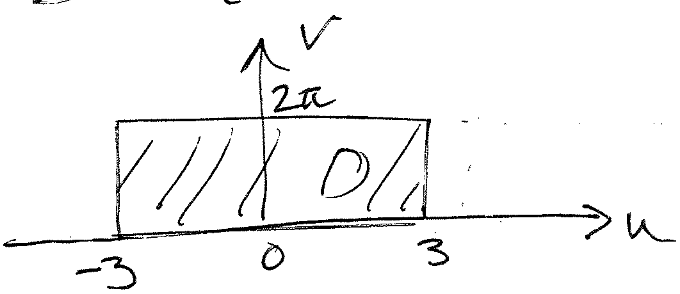
Angle $= v$

$D = \{0 \le v \le 2\pi,\ -3 \le y \le 3\}$



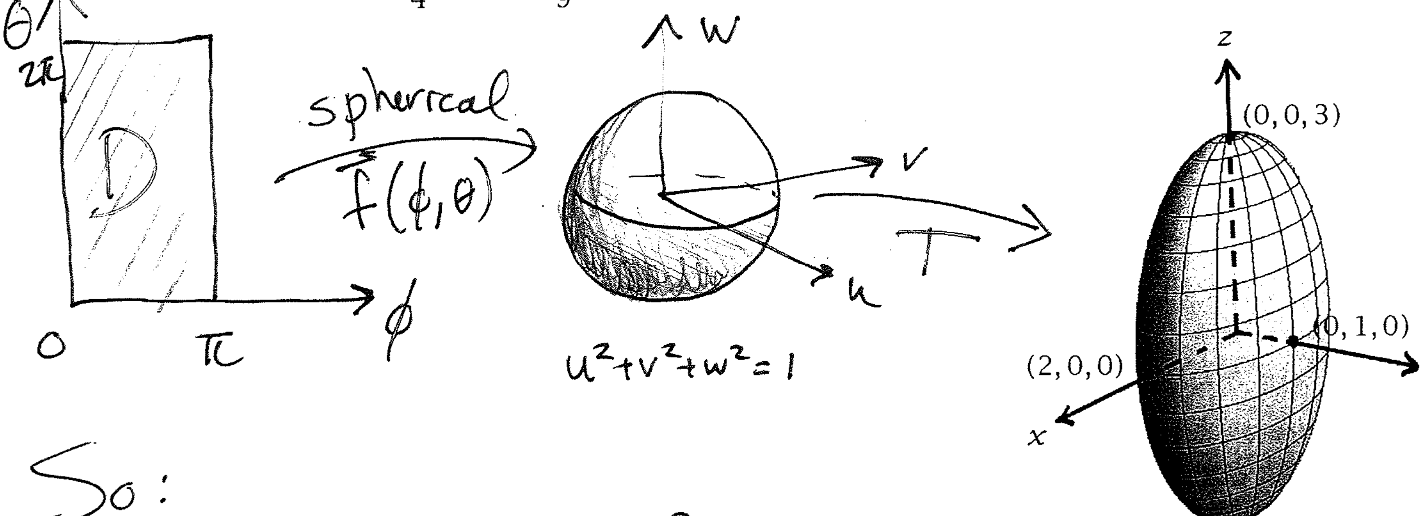
$x = 2\cos v$

$z = 2\sin v$



$\vec{r}(u,v) = (2\cos v,\ u,\ 2\sin v)$

(b) The ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 1$. **(3 points)**



So:

$D = \{0 \le \phi \le \pi,\ 0 \le \theta \le \pi\}$ $\quad T(u,v,w) = (2u,\ v,\ 3w)$

and

$\vec{r}(\phi,\theta) = T(\vec{f}(\phi,\theta)) = (2\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ 3\cos\phi)$