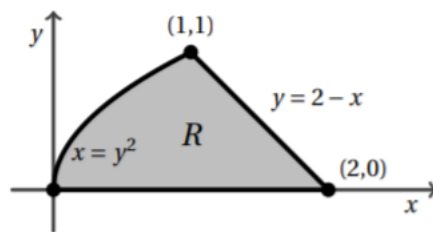


1. Let R denote the shaded region pictured below right. Compute $\iint_R 12y \, dA$. (4 points)



SOLUTION 1 (Integrate first with respect to x):

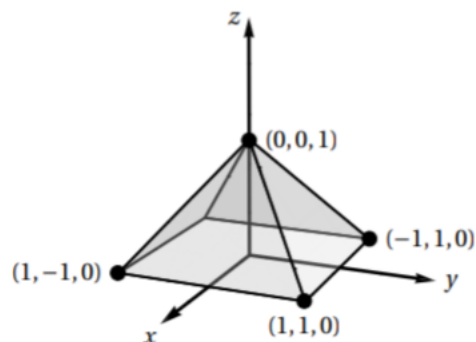
$$\begin{aligned} \iint_R 12y \, dA &= \int_0^1 \int_{y^2}^{2-y} 12y \, dx \, dy = \int_0^1 12xy \Big|_{x=y^2}^{x=2-y} dy = \int_0^1 12y(2-y-y^2) dy = \int_0^1 24y - 12y^2 - 12y^3 dy \\ &= (12y^2 - 4y^3 - 3y^4) \Big|_{y=0}^{y=1} = 12 - 4 - 3 = 5. \end{aligned}$$

SOLUTION 2 (Integrate first with respect to y):

$$\begin{aligned} \iint_R 12y \, dA &= \int_0^1 \int_0^{\sqrt{x}} 12y \, dy \, dx + \int_1^2 \int_0^{2-x} 12y \, dy \, dx = \int_0^1 6y^2 \Big|_{y=0}^{y=\sqrt{x}} dx + \int_1^2 6y^2 \Big|_{y=0}^{y=2-x} dx \\ &= \int_0^1 6x \, dx + \int_1^2 6(2-x)^2 \, dx = 3x^2 \Big|_0^1 - 2(2-x)^3 \Big|_1^2 \\ &= 3 + 2 = 5. \end{aligned}$$

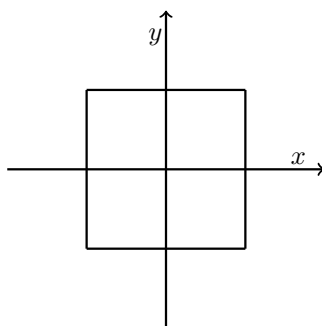
$\iint_R 12y \, dA = 5$

2. Completely setup, but do not evaluate, a triple integral giving the volume of the pyramid shown at right. This pyramid has a square base, and the triangular faces lie in the planes given by $x+z=1$, $-x+z=1$, $y+z=1$, and $-y+z=1$. (5 points)



SOLUTION:

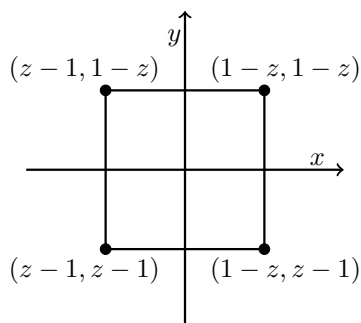
It is convenient to put the integral on the z -variable on the outside. This corresponds to slicing along the z -axis with planes parallel to the (x, y) -plane. For each z between 0 and 1, the cross section is a square with center on the z -axis. Its projection onto the (x, y) -plane looks like this:



The vertices of the square lie in the planes whose equations given above. Solving them for x and y gives

$$x = 1 - z, \quad x = z - 1, \quad y = 1 - z, \quad y = z - 1.$$

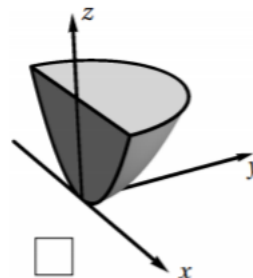
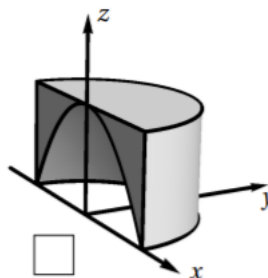
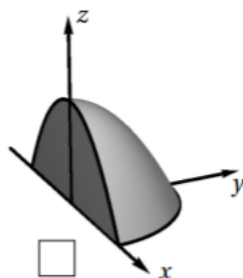
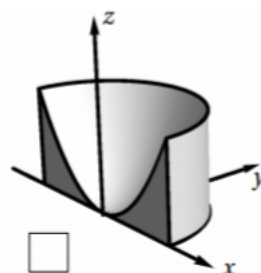
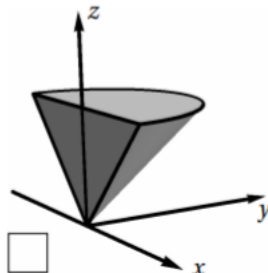
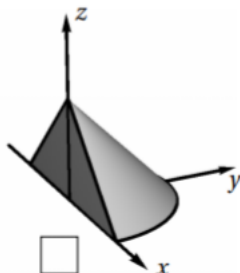
Note that for z between 0 and 1 we have $1 - z \geq 0$ and $z - 1 \leq 0$. Hence



For each fixed z , both x and y vary between $z - 1$ and $z - 1$.

$$Volume = \int_0^1 \int_{z-1}^{1-z} \int_{z-1}^{1-z} 1 \, dx \, dy \, dz$$

3. For each of the integrals: (a) $\int_0^\pi \int_0^1 \int_0^{r^2} f(r, \theta, z) r \, dz \, dr \, d\theta$ and (b) $\int_0^\pi \int_0^1 \int_0^{\sqrt{z}} f(r, \theta, z) r \, dr \, dz \, d\theta$ label the solid corresponding to the region of integration below. (2 points each)



SOLUTION:

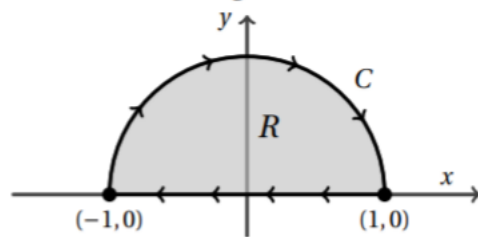
For reference, number the figures from 1 to 6 from top left to bottom right.

For (a), note that the height z goes from 0 to r^2 . The only figures where the height of the solid is bounded below by 0 are 1, 3 and 4. In 1 and 4, however, the height decreases with the radius. Hence (a) corresponds to figure 3.

For (b), note that at height z the radius of the solid goes from 0 to \sqrt{z} . The only two figures where the radius increases with the height are number 2 and number 6. In number 2, however, the radius grows linearly with the height. Hence the integral (b) corresponds to figure 6.

4. Let $\mathbf{F}(x, y) = \langle x - 1, \cos y + 2x - e^{y^2} \rangle$. Let R denote the solid semi-disk shown below right. Let C denote the boundary of the region R .

- (a) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C has the orientation shown. (3 points)



SOLUTION:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = - \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = - \iint_R 2 dA = -2\text{Area}(R) = -\pi.$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = -\pi}$$

- (b) Let D denote the part of the curve C above consisting *only* of the semicircle (*not* the line segment) with the orientation shown. Compute $\int_D \mathbf{F} \cdot d\mathbf{r}$. (3 points)

SOLUTION:

Let C'' denote the oriented segment from $(-1, 0)$ to $(1, 0)$. Then

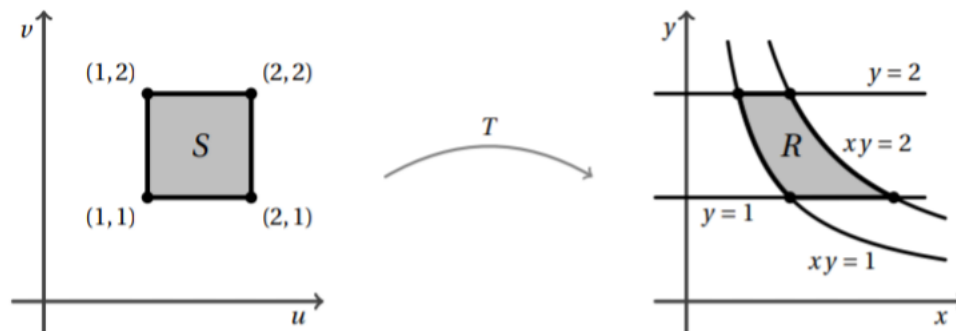
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} - \int_{C''} \mathbf{F} \cdot d\mathbf{r}$$

Using Part (a) we get

$$\int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C''} \mathbf{F} \cdot d\mathbf{r} = -\pi + \int_{C''} \mathbf{F} \cdot d\mathbf{r} = -\pi + \int_{-1}^1 P dx + Q dy = -\pi + \int_{-1}^1 (x - 1) dx = -\pi - 2.$$

$$\boxed{\int_{C'} \mathbf{F} \cdot d\mathbf{r} = -\pi - 2}$$

5. Let R be the region shown at right.



(a) Find a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ taking $S = [1, 2] \times [1, 2]$ to R . (3 points)

SOLUTION:

We want to find expressions for x and y in terms of u and v , so that $(x, y) = T(u, v)$. Note that the second coordinate is preserved, and therefore $y = v$. Moreover, looking at the two vertical sides of the square, we see that they are sent to hyperbolae. More precisely, the line $u = 1$ is sent to the hyperbola $xy = 1$, and the line $x = 2$ is sent to the hyperbola $xy = 2$. Hence we have $xy = u$. Since $y = v$, we conclude that $x = u/v$.

$$T(u, v) = \left\langle \frac{u}{v}, v \right\rangle$$

(b) Use your transformation $T(u, v)$ from part (a) to evaluate $\iint_R y^2 dA$ via an integral over S . (4 points)

Emergency backup transformation: if you can't do (a), pretend you got the answer $T(u, v) = \left(u^5 v, \frac{1}{u}\right)$ and do part (b) anyway.

SOLUTION:

The Jacobian matrix of T is given by

$$J = \begin{pmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{pmatrix}.$$

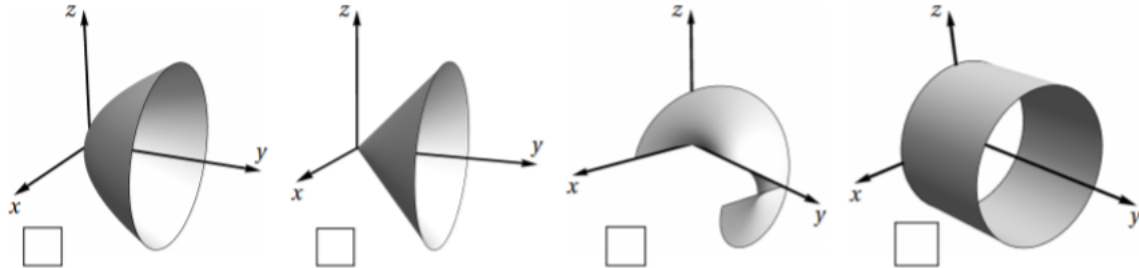
Hence $\det J = 1/v$. We have

$$\iint_R y^2 dA = \iint_S v^2 |\det J| du dv = \int_1^2 \int_1^2 v^2 \cdot \frac{1}{v} du dv = \int_1^2 v dv = \frac{v^2}{2} \Big|_{v=1}^{v=2} = 1.5$$

$$\iint_R y^2 dA = 1.5$$

6. Let S be the surface parameterized by $\mathbf{r}(u, v) = \langle v \cos u, v, v \sin u \rangle$ for $0 \leq u \leq 2\pi$ and $0 \leq v \leq 1$.

(a) Mark the picture of S below. (2 points)



SOLUTION:

The parameterization satisfies $x^2 + z^2 = y^2$, which is the equation of a cone. The correct picture is therefore the second one.

(b) Evaluate the surface integral $\iint_S y \, dS$. (6 points)

SOLUTION:

We start by computing $|\mathbf{r}_u \times \mathbf{r}_v|$. We have

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -v \sin u & 0 & v \cos u \\ \cos u & 1 & \sin u \end{vmatrix} = \langle -v \cos u, v, -v \sin u \rangle.$$

Hence

$$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{v^2 \cos^2 u + v^2 + v^2 \sin^2 u} = \sqrt{2}v.$$

The wanted integral is given by

$$\iint_S y \, dA = \int_0^{2\pi} \int_0^1 v |\mathbf{r}_u \times \mathbf{r}_v| \, dv \, du = \int_0^{2\pi} \int_0^1 \sqrt{2}v^2 \, dv \, du = \int_0^{2\pi} \left. \frac{\sqrt{2}}{3} v^3 \right|_{v=0}^{v=1} du = \int_0^{2\pi} \frac{\sqrt{2}}{3} \, du = \frac{2\sqrt{2}}{3} \pi.$$

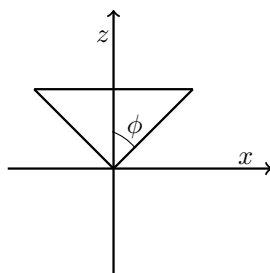
$$\boxed{\iint_S y \, dA = \frac{2\sqrt{2}}{3} \pi}$$

7. Let E be the solid above the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. Use **spherical coordinates** to completely setup, but not evaluate, a triple integral which computes the volume of E . (4 points)

SOLUTION:

We need to find bounds for the spherical coordinates θ , ϕ , and ρ . Clearly we have $0 \leq \theta \leq 2\pi$.

To find the bounds for ϕ , we look at the intersection of E with the plane $y = 0$. This intersection consists of the triangle enclosed by the lines $z = 2$, $z = x$, and $z = -x$.



Hence $0 \leq \phi \leq \pi/4$.

To find the bounds on ρ , recall that $z = \rho \cos \phi$. The maximum value of z in E is $z = 2$. Hence the upper bound for ρ is when $2 = \rho \cos \phi$, that is, $\rho = 2 \sec \phi$.

Finally, recall that in spherical coordinates we have $dV = \rho^2 \sin \phi d\rho d\phi d\theta$.

$$Volume = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2 \sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta$$

8. For each surface S in parts (a) and (b), give a parameterization $\mathbf{r}: D \rightarrow S$. Be sure to explicitly specify the domain D and call your parameters u and v .

(a) The portion of the sphere $x^2 + y^2 + z^2 = 1$ where $x \geq 0$. **(3 points)**

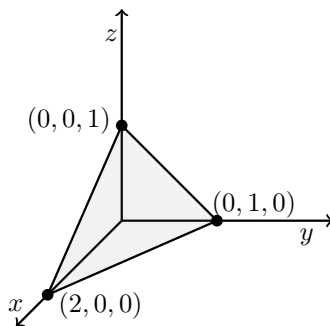
SOLUTION: We use spherical coordinates with $\rho = 1$. We take as parameters $u = \phi$ and $v = \theta$. Note that we are trying to parameterize the half of the unit sphere for which $x \geq 0$. In this region $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. As in the parameterization of the sphere, ϕ varies between 0 and π .

$$D = \left\{ 0 \leq u \leq \pi \text{ and } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}$$

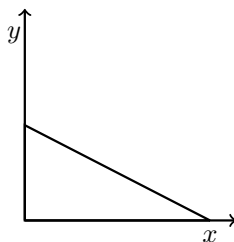
$$\mathbf{r}(u, v) = \{\sin u \cos v, \sin u \sin v, \cos u\}$$

(b) The triangle in \mathbb{R}^3 with vertices $(2, 0, 0), (0, 1, 0), (0, 0, 1)$ which lies in the plane $\frac{x}{2} + y + z = 1$. (2 points)

SOLUTION: We can use $u = x$ and $y = v$. Then $z = 1 - x/2 - y = 1 - u/2 - v$. Ww now need to find bounds for x and y . We are trying to parameterize the triangle shown in the figure below.



The region to parameterize lies over the triangle on the (x, y) plane represented below. This triangle has vertices in $(0, 0), (2, 0)$ and $(0, 1)$. The hypotenuse is part of the line $x/2 + y = 1$ (intersection of $x/2 + y + z = 1$ with $z = 0$).



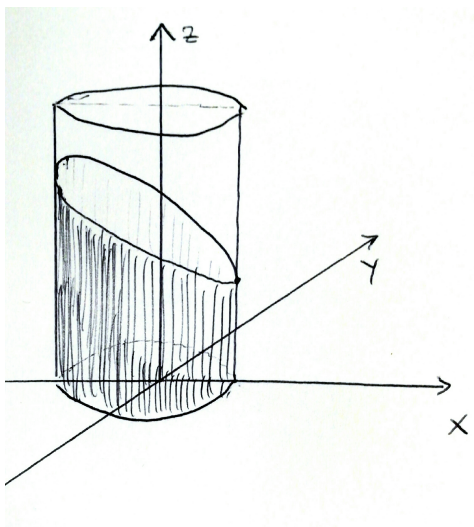
Note that x varies between 0 and 2. For each such x , the variable y varies between 0 and $1 - x/2 - y$.

$$D = \left\{ 0 \leq u \leq 2 \text{ and } 0 \leq 1 - \frac{u}{2} \right\}$$

$$\mathbf{r}(u, v) = \left\{ u, v, 1 - \frac{u}{2} - v \right\}$$

- (c) Parameterize the cylinder $C = \{x^2 + y^2 = 1\}$ by $\mathbf{r}(u, v) = \langle \cos u, \sin u, v \rangle$ for $0 \leq u \leq 2\pi$ and v unrestricted. Let M be the part of C above the xy -plane and below the plane $x + z = 2$. Find a region D in \mathbb{R}^2 so that $\mathbf{r}(D) = M$. **(1 point)**

SOLUTION: The region M is the portion of the cylinder shaded in the figure below.



At the bottom of M we have $z = v = 0$.

At the top of M we have $x + z = 2$, that is, $\cos u + z = 2$. Hence the upper bound for v is $v = 2 - \cos u$.

$$D = \{0 \leq u \leq 2\pi \text{ and } 0 \leq v \leq 2 - \cos u\}$$

- (d) Let M be the surface in part (c). Is the surface integral $\iint_M x \, dS$:

negative zero positive

Circle your answer. **(1 point)**

SOLUTION: Looking at the picture above, we see that for the largest portion of the surface M the x -coordinate is negative.

$$\iint_M x \, dS \text{ is NEGATIVE}$$