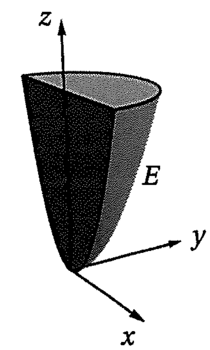
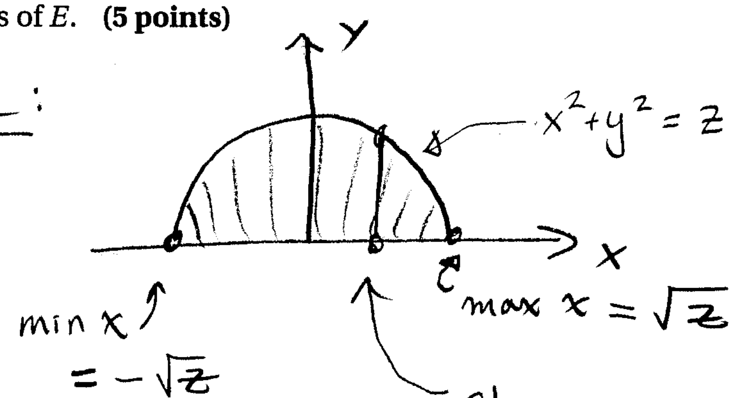
1. Suppose the region E where $x^2 + y^2 \le z \le 4$ and $y \ge 0$ is made of material whose density is given by $\rho(x, y, z) = z$.



(a) Fill in the limits and integrand of the integral below so that it computes the mass of E. **(5 points)**

z - slice :



slice once x is also fixed

$y_{max} = \sqrt{z - x^2}$

$$\boxed{\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} z \quad dy\,dx\,dz}$$

(b) Circle the center of mass of E, whose coordinates have been rounded to one decimal place. Note: This can be done without evaluating any integrals. **(2 points)**

(0, 0.7, 1)	(0, 0.7, 2)	(0, 0.7, 3)	(0.7, 0, 1)	(0.7, 0, 2)	(0.7, 0, 3)

Scratch Space

By symmetry, the x-component of the center of mass is 0. Since the density increases as you go up, and more than 1/2 of the volume is above the plane $z = 2$, the z-component of the center of mass must be > 2 and so the correct answer must be

(0, 0.7, 3)

2. Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(u, v) = \left(u^2 - v^2,\ uv\right)$.

(a) Circle the Jacobian of T: **(2 points)**

$$\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix} = 2u^2 + 2v^2$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \boxed{2u^2+2v^2 \quad 2u^2-2v^2 \quad 4uv \quad 2u+2v}$$

(b) Let S be the square in the (u, v)–plane where $0 \le u \le 1$ and $0 \le v \le 1$. Find the area of its image $T(S)$ in the (x, y)–plane. **(3 points)**

$$\text{Area} = \iint_{T(S)} 1\, dA = \iint_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= 2\int_0^1\int_0^1 u^2+v^2\, du\,dv = 2\int_0^1 \frac{u^3}{3} + v^2 u\Big|_{u=0}^{u=1} dv$$

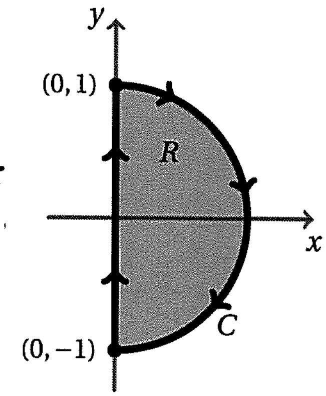
$$= 2\int_0^1 \frac{1}{3} + v^2\, dv = 2\left(\frac{1}{3}v + \frac{1}{3}v^3\Big|_{v=0}^{1}\right) = 4/3$$

$$\boxed{\text{Area}(T(S)) = 4/3}$$

Scratch Space

3. Consider the vector field $\mathbf{F}(x, y) = \langle ye^x,\ e^x + x\rangle$. Let R be the half disk below, and let C be the boundary of R, oriented as shown.

$P \qquad Q$



(a) Use Green's Theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(4 points)**

$$\int_C \vec{F} \cdot dr = -\int_{-C} \vec{F} \cdot d\vec{r} = -\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= -\iint_R (e^x + 1) - (e^x)\, dA = -\iint_R 1\, dA$$

$$= -\text{Area}(A) = -\pi/2$$

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = -\pi/2}$$

(b) Let C_0 be the round part of C, that is, just the semicircle from $(1,0)$ to $(-1,0)$, not including the y-axis. Compute $\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$. **(2 points)** Let A be the vertical segment from $(0,-1)$ to $(0,1)$. Then $\int_C \vec{F} \cdot dr = \int_{C_0} \vec{F} \cdot dr + \int_A \vec{F} \cdot d\vec{r}$.

Now $\int_A \vec{F} \cdot dr = \int_{-1}^{1} \langle y, 1\rangle \cdot \langle 0, 1\rangle\, dt = \int_{-1}^{1} 1\, dt = 2$

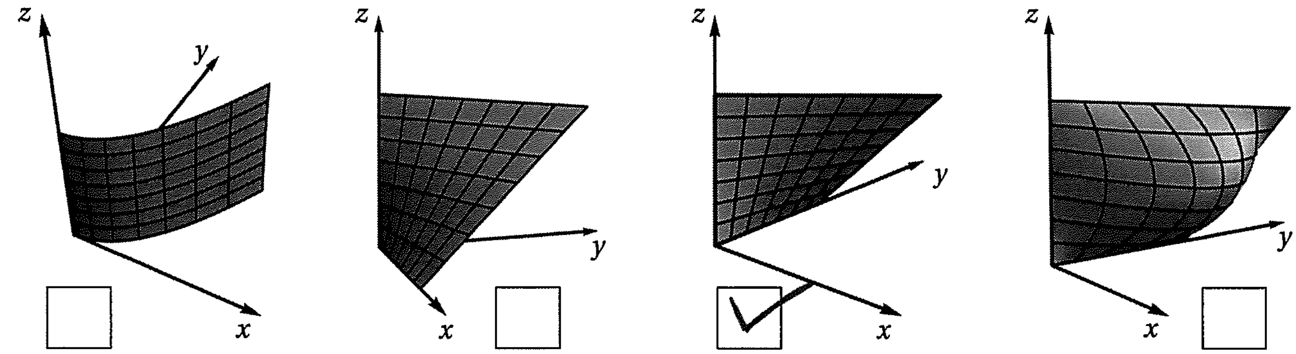
where we have used the param. $\vec{r}(t) = (0, t)$ for $-1 \leq t \leq 1$. Thus:

$$\boxed{\int_{C_0} \mathbf{F} \cdot d\mathbf{r} = -\pi/2 - 2}$$

Scratch Space

4. Let S be the surface parameterized by $\mathbf{r}(u,v) = \langle uv,\ u,\ v\rangle$ for $0 \le u \le 1$ and $0 \le v \le 1$.

(a) Mark the box next to the picture of S below. **(2 points)**



(b) Find a normal vector $\mathbf{v}$ for the tangent plane to S at the point $\left(\frac{1}{4}, \frac{1}{2}, \frac{1}{2}\right)$. **(3 points)**

Have $\vec{r}_u = \langle v, 1, 0\rangle$ $\vec{r}_v = \langle u, 0, 1\rangle$ and so

$$\vec{v} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v & 1 & 0 \\ u & 0 & 1 \end{vmatrix} = \langle 1, -v, -u\rangle$$

Our point is where $u = v = 1/2$, so plugging in gives

$\boxed{\mathbf{v} = \langle 1, -1/2, -1/2\rangle}$

(c) Completely set up, but do not evaluate, the surface integral $\iint_S x + z\, dS$. **(3 points)**

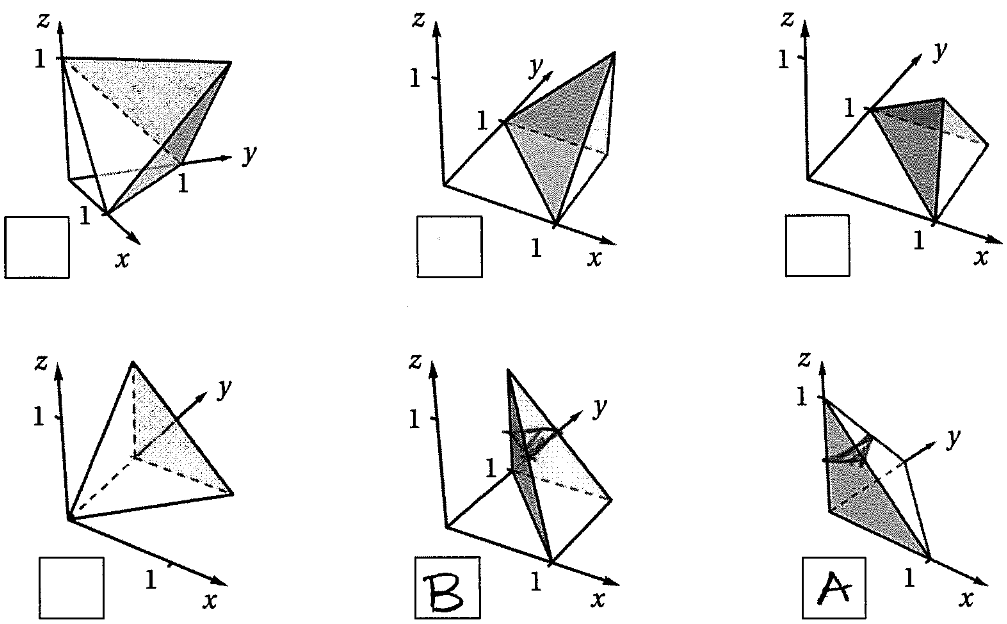
$$\iint_S x+z\, dS = \int_0^1 \int_0^1 (uv+v)\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

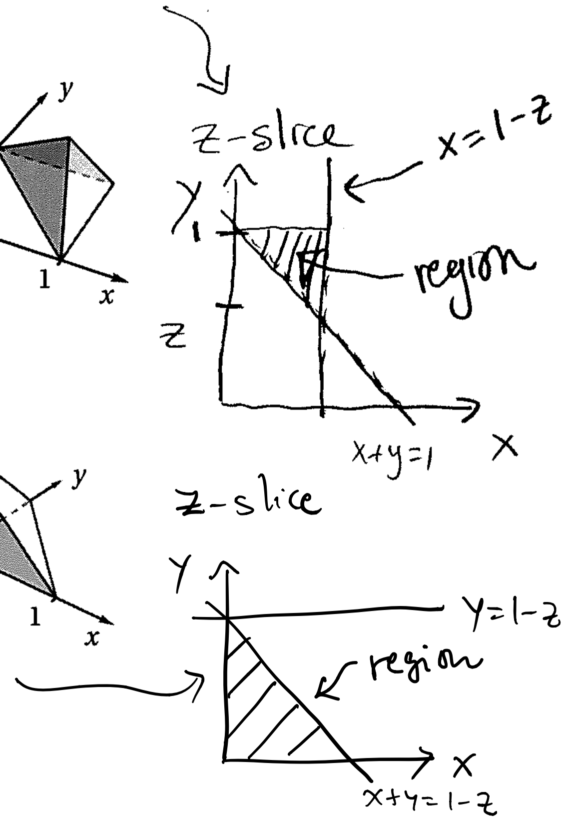
$$= \int_0^1 \int_0^1 (uv+v)\sqrt{1+u^2+v^2}\, du\, dv$$

$$\boxed{\int_0^1 \int_0^1 (uv+v)\sqrt{1+u^2+v^2}\, du\, dv}$$

5. Label the boxes next to the solid regions corresponding to the following two integrals: **(2 points each)**

(A) $\displaystyle\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f(x,y,z)\, dx\, dy\, dz$ (B) $\displaystyle\int_0^1 \int_z^1 \int_{1-y}^{1-z} g(x,y,z)\, dx\, dy\, dz$





6. Consider the solid $E = \{x^2 + y^2 + z^2 \le 4 \quad \text{and} \quad x \le 0 \quad \text{and} \quad z \le 0\}$.

(a) Check the box next to the correct description of E in terms of spherical coordinates: **(2 points)**

- $\{0 \le \rho \le 2,\ 0 \le \theta \le \pi,\ \pi/2 \le \phi \le \pi\}$
- $\{0 \le \rho \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$
- $\{0 \le \rho \le 2,\ 0 \le \theta \le \pi,\ 0 \le \phi \le \pi/2\}$
- ✓ $\{0 \le \rho \le 2,\ \pi/2 \le \theta \le 3\pi/2,\ \pi/2 \le \phi \le \pi\}$
- $\{0 \le \rho \le 2,\ \pi/2 \le \theta \le 3\pi/2,\ 0 \le \phi \le \pi/2\}$

(b) Select the correct integrand that fills in the blank of $\iiint_E z\, dV = \iiint_E$ _________ $d\rho\, d\theta\, d\phi$. **(2 points)**

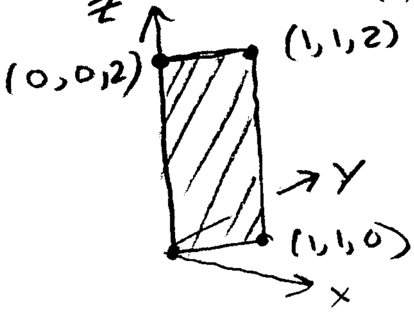
- $\rho^2 \sin\phi \cos\phi$
- $\rho^3 \sin\phi \sin\theta$
- $\rho^2 \sin\phi$
- $\rho \cos^2\theta$
- ✓ $\rho^3 \sin\phi \cos\phi$
- $\rho \sin\phi \cos\theta$

4 (a). Setting $v = a$, where a is a fixed constant, corresponds to taking a horizontal slice of S given by $\langle ua, u, a \rangle$, which is a line, so the first and fourth options are out. The second option is out because the x-axis $(y = z = 0)$ intersects S only at the origin.

6 (a). Since $z \le 0$, $\rho \cos\phi \le 0$, so $\cos\phi \le 0$, hence $\pi/2 \le \phi \le \pi$. Since $x \le 0$, $\rho \sin\phi \cos\theta \le 0$. Since $\sin\phi \ge 0$ (this is always true), we must have $\cos\theta \le 0$, i.e $\pi/2 \le \theta \le 3\pi/2$. Finally, $0 \le \rho \le 2$ follows from $x^2 + y^2 + z^2 \le 4$.

6(b). In polar coordinates, $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Since $z = \rho\cos\phi$, we have $z\,dV = \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta$.

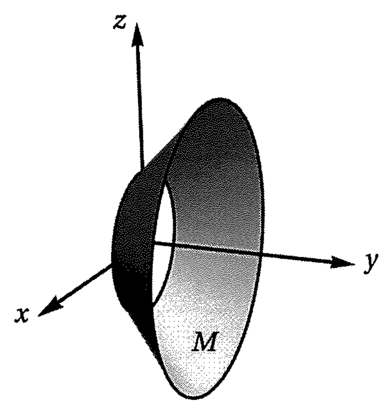
7. (a) Consider the rectangle S in $\mathbb{R}^3$ with vertices $(0,0,0)$, $(1,1,0)$, $(1,1,2)$, and $(0,0,2)$. Give a parameterization of S of the form $\mathbf{r}(u,v)$ where $0 \le u \le 1$ and $0 \le v \le 1$. **(2 points)**



This is the plane $x=y$ in the region $0 \le x \le 1$, $0 \le z \le 2$.

$\mathbf{r}(u,v) = \langle \; u \; , \; u \; , \; 2v \; \rangle$

(b) Let M be the portion of the cone $\sqrt{x^2+z^2} = y+1$ for $0 \le y \le 1$ as shown at right. Parameterize it by $\mathbf{r}\colon D \to \mathbb{R}^3$, being sure to specify the domain D of the parameterization in the (u,v)–plane. **(3 points)**

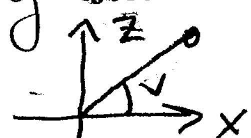


Ans 1: Take parameters $(u, v) = (x, z)$

Then $\vec{r}(u,v) = \left(u, 1 - \sqrt{x^2+z^2}, v\right)$

and $D = \{ 1 \le u^2+v^2 \le 2 \}$

Ans 2: Take parameters $u = y$ and v = angle around the y-axis:



For u fixed the corresponding circle on M has radius $r = u+1$. Thus:

$\mathbf{r}(u,v) = \langle (u+1)\cos v \; , \; u \; , \; (u+1)\sin v \rangle$

$D = \{ 0 \le u \le 1 \text{ and } 0 \le v \le 2\pi \}$

(c) The surface integral $\iint_M z^2 \, dS$ is: negative zero positive **(1 point)**

Scratch Space

The integrand is always ≥ 0 and is > 0 most places. So the integral must be positive.