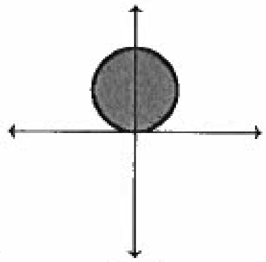
1. Suppose the integral of the function $f(x, y) = x^2 + y$ over a region R has the following form after changing to polar coordinates:

$$\iint_R x^2 + y \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} g(r,\theta)\, dr\, d\theta,$$

for some function $g(r,\theta)$.

(a) Which of the following shows the region R in the xy-plane? **(2 points)**



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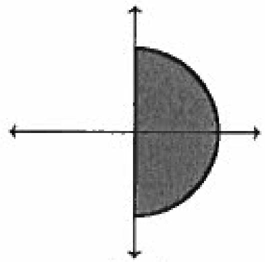

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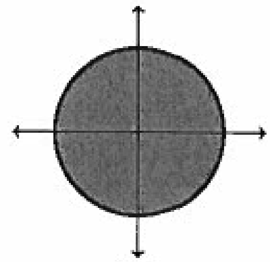

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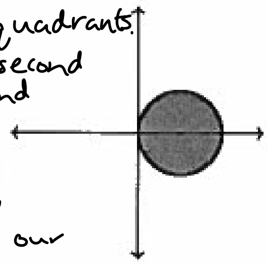
Explanation of (a):

The limits on θ tell us the region should be contained only in the 1st and 4th quadrants. This eliminates all but the second and fourth regions. The second region has radius from 0 to some constant, but our limits go from 0 to $2\cos\theta$, so the second region is not our region. Thus, the region must be the fourth region.



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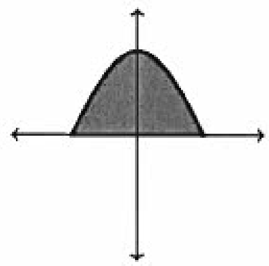

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(b) Find the integrand $g(r,\theta)$ and fill in blank in the integral below. **(3 points)**

$$\iint_R f(x,y)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)\, r \, dr\, d\theta$$

$$\Rightarrow \quad g(r,\theta) = f(r\cos\theta, r\sin\theta)\, r$$

$$= \left[(r\cos\theta)^2 + (r\sin\theta)\right] r$$

$$= r^3\cos^2\theta + r^2\sin\theta$$

$$\iint_R x^2 + y\, dA = \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^3\cos^2\theta + r^2\sin\theta \quad dr\, d\theta}$$

2. Compute the following double integral: **(5 points)**

$$\int_1^2 \int_0^x 2x + 2y \, dydx.$$

$\int_1^2 \left(\int_0^x (2x + 2y) \, dy \right) dx = \int_1^2 \left[2xy + y^2 \right]_0^x \, dx$

$= \int_1^1 (2x^2 + x^2) - (0) \, dx = \int_1^2 3x^2 \, dx$

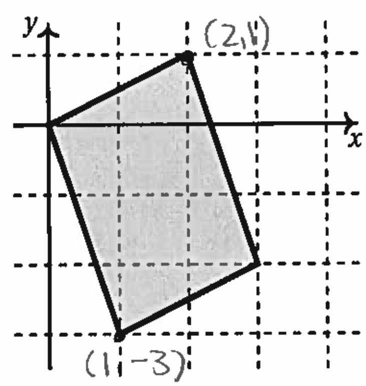
$= \left[x^3 \right]_1^2$

$= 8 - 1$

$= 7$

$\int_1^2 \int_0^x 2x + 2y \, dydx =$ $\boxed{7}$

3. Find a <u>linear transformation</u> T that sends the unit square $[0,1] \times [0,1]$ to the parallelogram drawn to the right against a dashed grid of unit squares. **(3 points)**



<u>Option 1</u> $T(1,0) = (1,-3) \Rightarrow T(u,0) = (u, -3u)$

$T(0,1) = (2,1) \Rightarrow T(0,v) = (2v, v)$

$\Rightarrow \boxed{T(u,v) = (u + 2v, \; -3u + v)}$

<u>Option 2:</u> $T(1,0) = (2,1) \Rightarrow T(u,0) = (2u, u)$

$T(0,1) = (1,-3) \Rightarrow T(0,v) = (v, -3v)$

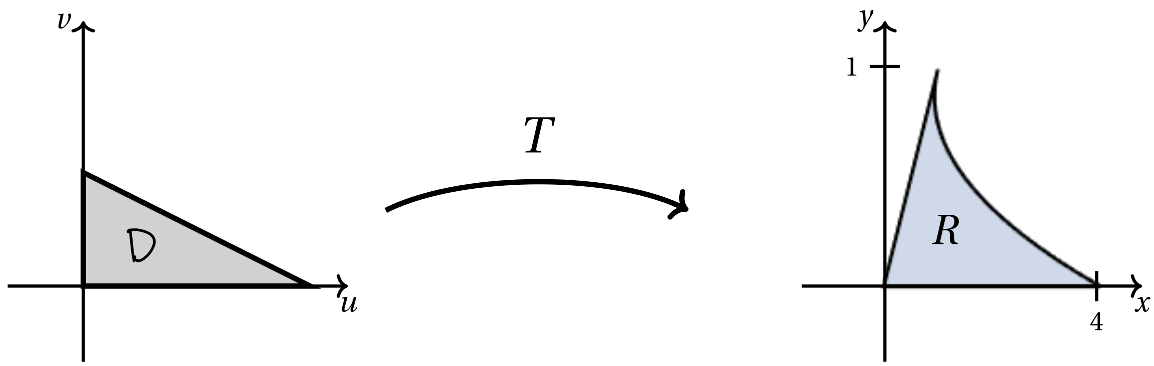
$\boxed{\Rightarrow T(u,v) = (2u + v, u - 3v)}$

$T(u, v) = \boxed{(\quad , \quad)}$

<u>Note:</u> We are using the following fact about linear transformations:

$T(u,v) = T(u,0) + T(0,v) = uT(1,0) + vT(0,1).$

4. The transformation $T(u,v) = (u^2+v, v)$ takes the triangle with vertices $(0,0)$, $(2,0)$, and $(0,1)$ to the region R as shown below. Find the missing limits of integration and circle the correct integrand for the iterated integral that computes *the area of* R. (Note that the order of integration is already determined.) **(5 points)**

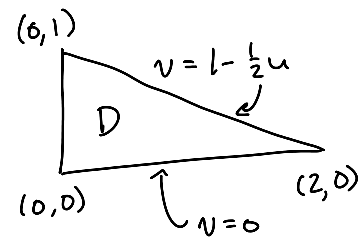


$$\text{Area }(R) = \iint_R 1\, dA = \iint_{T(D)} 1\, dA = \iint_D |\det J|\, dA,$$ where J is the Jacobian of the transformation T.

Limits:

We are given that the order of integration is $dv\,du$. Based on the region D, v is bounded by the functions $v=0$ and $v = 1-\frac{1}{2}u$. We also see that $0 \le u \le 2$. So our limits are

$$0 \le v \le 1-\tfrac{1}{2}u,$$
$$0 \le u \le 2.$$



Integrand:

The integrand is $g(u,v) = |\det J|$.

$$\det J = \begin{vmatrix} \frac{\partial}{\partial u}(u^2+v) & \frac{\partial}{\partial v}(u^2+v) \\ \frac{\partial}{\partial u}(v) & \frac{\partial}{\partial v}(v) \end{vmatrix} = \begin{vmatrix} 2u & 1 \\ 0 & 1 \end{vmatrix} = 2u.$$

Then

$$g(u,v) = |\det J| = |2u| = 2u.$$

Note that $|2u| = 2u$ in our region because $0 \le u \le 2$, i.e. u is non-negative.

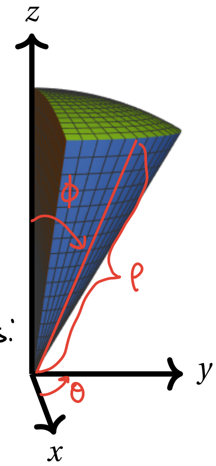
Fill in the limits of integration

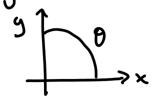
$$\text{Area}(R) = \boxed{\int_0^2 \int_0^{1-\frac{1}{2}u} g(u,v)\, dv\, du}$$

Circle the correct integrand. $g(u,v) =$

$u^2 v$	uv^2	$2u$	$2v$	u^2+v	v^2+u

5. In this problem you are to set up the integral of the function yz over the region R (shown to the right) in the first octant above the cone $z^2 = 3x^2 + 3y^2$ and inside the sphere $x^2 + y^2 + z^2 = 16$. Check the box next to the integral with the correct limits of integration, then circle the correct integrand. **(4 points)**

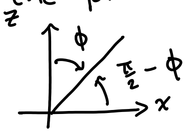


- inside sphere $x^2+y^2+z^2=16 \Rightarrow 0 \le \rho \le \sqrt{16} = 4$.
- first octant  $\Rightarrow 0 \le \theta \le \pi/2$.

- To find ϕ, we must determine the angle made between the cone and the z-axis. To do this, we can write the cone equation in spherical coordinates:

$$\rho^2 \cos^2\phi = 3\rho^2 \sin^2\phi \cos^2\theta + 3\rho^2 \sin^2\phi \sin^2\theta$$
$$\rho^2 \cos^2\phi = 3\rho^2 \sin^2\phi$$
$$\cos^2\phi = 3\sin^2\phi \Longrightarrow \phi = \pi/6. \text{ So } 0 \le \phi \le \pi/6.$$

Alternatively, we can determine ϕ by looking at the intersection of the cone with the plane $y=0$. When $y=0$, our cone equation becomes $z^2 = 3x^2 \Rightarrow z = \sqrt{3}x$, (we can just take the positive square root since we're in the first octant). Then,



$$\tan(\pi/2 - \phi) = \frac{z}{x} = \frac{\sqrt{3}x}{x} = \sqrt{3} \Rightarrow \pi/2 - \phi = \frac{\pi}{3} \Rightarrow \phi = \frac{\pi}{6}.$$

So again we get $0 \le \phi \le \pi/6$.

$$\iiint_R yz\, dV =$$

- If $g(x,y,z) = yz$, then our integrand is

$$h(\rho,\theta,\phi)\rho^2\sin\phi = g(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi$$
$$= (\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi) = \rho^4\sin\theta\sin^2\phi\cos\phi.$$

☐ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \int_0^2 h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$ ☐ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_0^2 h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$

☒ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \int_0^4 h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$ ☐ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_0^4 h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$

☐ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{6}} \int_0^{16} h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$ ☐ $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_0^{16} h(\rho,\theta,\phi)\, d\rho\, d\phi\, d\theta$

$h(\rho,\theta,\phi) =$

$\rho^3\sin\theta\sin^2\phi\cos\phi$	$\rho^4\sin\theta\sin^2\phi\cos\phi$	$\rho^5\sin\theta\cos^2\phi\sin\phi$
$\rho^3\cos\theta\sin^2\phi\cos\phi$	$\rho^4\cos\theta\sin^2\phi\cos\phi$	$\rho^5\cos\theta\cos^2\phi\sin\phi$

6. Consider a solid object with density ρ_0 that occupies a region E in $\mathbf{R}^3$. The moment of inertia of the solid around the x-axis is given by

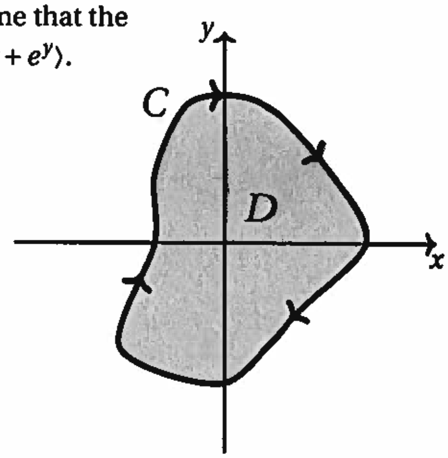
$$I_x = \iiint_E g(x,y,z)\, dV,$$

for some function $g(x,y,z)$. Circle the correct integrand. **(2 points)**

$g(x,y,z) =$ | $\rho_0(x^2+y^2+z^2)$ | $\rho_0(y^2+z^2)$ | $\rho_0(x^2+y^2)$ | $\rho_0(x^2+z^2)$ | $\rho_0 x^2$ | $\rho_0 x$ | $\rho_0 y^2$ | $\rho_0 y$ |

$$I_x = \iiint_E (\text{distance to } x\text{-axis})^2(\text{density})\, dV = \iiint_E (y^2+z^2)\rho_0\, dV$$

7. Let C be the oriented curve shown at right, oriented **clockwise**. Assume that the region D bounded by C has area 6. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = \langle \sin x, 2x + e^y\rangle$. **(4 points)**



By Green's theorem, since $C = -\partial D$

$$\int_C \vec{F}\cdot d\vec{r} = -\iint_D (Q_x - P_y)\,dA$$

$$= -\iint_D 2\,dA$$

$$= -2\iint_D dA$$

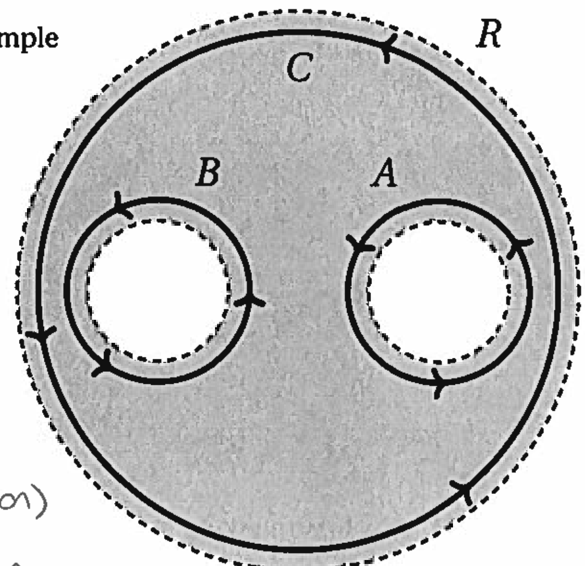
$$= -2\,(\text{Area}(D))$$

$$= -12$$

$\int_C \mathbf{F}\cdot d\mathbf{r} =$ $\boxed{-12}$

8. Consider the region R shown at the right which contains simple closed curves A, B, C, all oriented counterclockwise. Suppose that $\mathbf{F} = \langle P, Q\rangle$ is a vector field with continuous partial derivatives on R with the properties

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \int_A \mathbf{F}\cdot d\mathbf{r} = 2 \quad \text{and} \quad \int_B \mathbf{F}\cdot d\mathbf{r} = -1$$



(a) Find $\int_C \mathbf{F}\cdot d\mathbf{r}$. Circle your answer below. **(2 points)**

Let D be the region enclosed by C, outside of B and A

$\Rightarrow \partial D = -A - B + C$ (standard orientation)

By Green's theorem, $\int_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D Q_x - P_y\,dA = 0$

$\Rightarrow 0 = -\int_A \vec{F}\cdot d\vec{r} - \int_B \vec{F}\cdot d\vec{r} + \int_C \vec{F}\,d\vec{r}$

$= -2 + 1 + \int_C \vec{F}\,d\vec{r}$

$\int_C \mathbf{F}\cdot d\mathbf{r} =$	-3	-2	-1	0	1	2	3

(b) **True or False?** (Circle your answer.) The vector field **F** is conservative. **(2 points)**

True	False

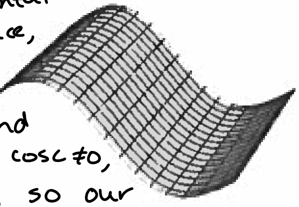
- If $\vec{F}$ were conservative, then $\int_{C_1} \vec{F}\cdot d\vec{r} = 0$ whenever C_1 is a closed curve.
- In our case, C is a closed curve, but (a) tells us $\int_C \vec{F}\cdot d\vec{r} = 1 \neq 0$, so $\vec{F}$ cannot be conservative.

9. Let S be the surface in $\mathbf{R}^3$ parameterized by $\mathbf{r}(u,v) = \langle v\cos u, v\sin u, u\rangle$, for u in $[0,2\pi]$ and v in $[-2,2]$.

(a) Mark the box below the picture of S shown here. **(2 points)**

Explanation of (a):

Consider a horizontal slice of our surface, i.e. fix $z=c$ for some constant c. Then $x = v\cos c$ and $y = v\sin c$. When $\cos c \neq 0$, we get $\frac{x}{\cos c} = v$, so our horizontal slices are lines of the form $y = \tan c\, x$, (or of the form $x = \cot c\, y$ when $\sin c \neq 0$). The only surface here whose horizontal slices are always lines through the origin is the second surface.



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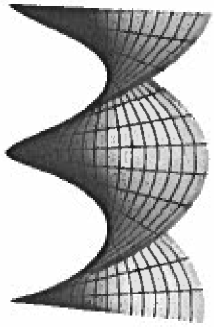

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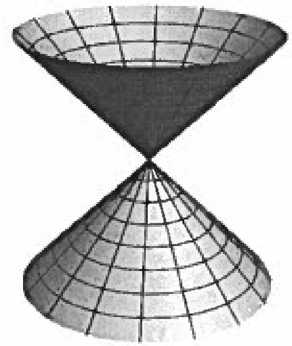

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(b) Compute the partial derivatives of $\mathbf{r}$. **(1 point each)**

$r_u = \langle \frac{\partial}{\partial u}(v\cos u), \frac{\partial}{\partial u}(v\sin u), \frac{\partial}{\partial u}(u)\rangle =$ $\mathbf{r}_u = \boxed{\langle -v\sin u, \; v\cos u, \; 1 \rangle}$

$r_v = \langle \frac{\partial}{\partial v}(v\cos u), \frac{\partial}{\partial v}(v\sin u), \frac{\partial}{\partial v}(u)\rangle =$ $\mathbf{r}_v = \boxed{\langle \cos u, \; \sin u, \; 0 \rangle}$

(c) Find a vector $\mathbf{n}$ that is perpendicular to the tangent plane to S at the point $\mathbf{r}\left(\frac{\pi}{6}, 1\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}, \frac{\pi}{6}\right)$. **(2 points)**

take $\vec{n} = |\vec{r}_u(P) \times \vec{r}_v(P)|$, $P = (\pi/6, 1)$

$\vec{r}_u(\pi/6, 1) = \langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 1\rangle$, $\vec{r}_v(\pi/6, 1) = \langle \frac{\sqrt{3}}{2}, \frac{1}{2}, 0\rangle$

$$\Rightarrow \vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1/2 & \sqrt{3}/2 & 1 \\ \sqrt{3}/2 & 1/2 & 0 \end{vmatrix} = \langle -1/2, \sqrt{3}/2, -\frac{1}{4} - 3/4 \rangle$$

$\mathbf{n} = \boxed{\langle -\frac{1}{2}, \; \frac{\sqrt{3}}{2}, \; -1 \rangle}$

(d) Fill in the limits and integrand of the double integral below that computes $\iint_S 2y\, dS$. Do not evaluate. **(3 points)**

$$\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA \qquad D = \begin{cases} 0 \le u \le 2\pi \\ -2 \le v \le 2 \end{cases}$$

$$\cdot\ \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 1 \\ \cos u & \sin u & 0 \end{vmatrix}$$

$$\int_S 2y\, dS = \boxed{\int_{-2}^{2} \int_0^{2\pi} 2v\sin u \sqrt{1+v^2}\, du\, dv}$$

$= \langle -\sin u, \cos u, -v\rangle$

$\Rightarrow |\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 u + \cos^2 u + v^2} = \sqrt{1+v^2}$

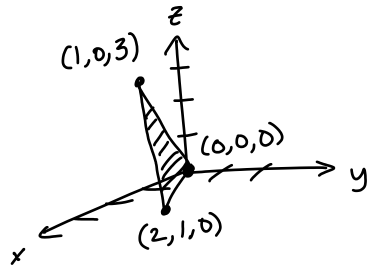
$\Rightarrow f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v| = 2v\sin u \sqrt{1+v^2}$

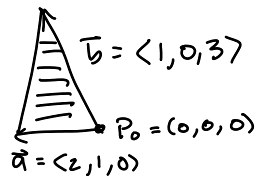
10. Let S be the triangle through the points $(0,0,0)$, $(2,1,0)$, and $(1,0,3)$. Parameterize the surface S by $\mathbf{r}\colon D \to \mathbf{R}^3$, being sure to specify the domain D of the parameterization in the (u,v) plane.
Note: you will receive partial credit if you parameterize the plane through those three points. **(4 points)**

In general, the vector equation for a plane can be written as

$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

where $\vec{r}_0$ is the position vector of a point P_0 in the plane, and $\vec{a}, \vec{b}$ are vectors in the plane.





In our case, we can take $P_0 = (0,0,0)$, so $\vec{r}_0 = \langle 0,0,0\rangle$, and we can choose $\vec{a} = \langle 2,1,0\rangle$, $\vec{b} = \langle 1,0,3\rangle$.

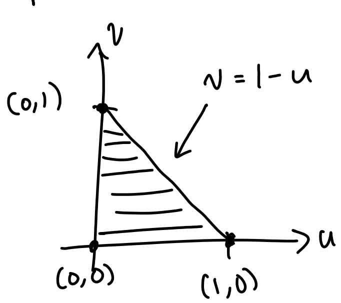
Thus, the parametrization of the plane containing our triangle is

$$\vec{r}(u,v) = u\vec{a} + v\vec{b} = u\langle 2,1,0\rangle + v\langle 1,0,3\rangle = \langle 2u+v, u, 3v\rangle.$$

To parametrize the triangle, we must determine the bounds of u and v.
Notice that

$$\vec{r}(0,0) = \langle 0,0,0\rangle, \quad \vec{r}(1,0) = \langle 2,1,0\rangle = \vec{a}, \quad \vec{r}(0,1) = \langle 1,0,3\rangle = \vec{b},$$

so the preimage of our triangle in the uv-plane is the triangle pictured below.



By this picture, the bounds we should choose are

$$0 \le u \le 1$$
$$0 \le v \le 1-u.$$

$$D = \boxed{\{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1-u\}}$$

$$\mathbf{r}(t) = \boxed{\langle 2u+v,\ u,\ 3v\rangle}.$$

Note:
- We could have chosen $\vec{a} = \langle 1,0,3\rangle$ and $\vec{b} = \langle 2,1,0\rangle$ instead.
- We could have chosen our limits to be $0 \le u \le 1-v$, $0 \le v \le 1$ instead.

All these changes would still give valid parametrizations.