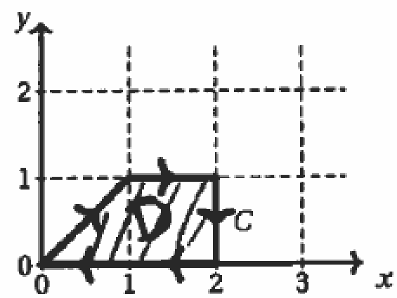
Question 1

. Let C be the oriented curve shown at right against a dashed grid of unit squares. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for $\mathbf{F} = \langle y^2+\cos x,\ x+e^y\rangle$. **(6 points)**

P Q



$\int_C \mathbf{F}\cdot d\mathbf{r} = -1/6$

By Mr. Green:

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$$

$-C \leftarrow$ counter-clockwise

$$= -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= \iint_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\, dA = \iint_D 2y-1\, dA$$

$$2y - 1$$

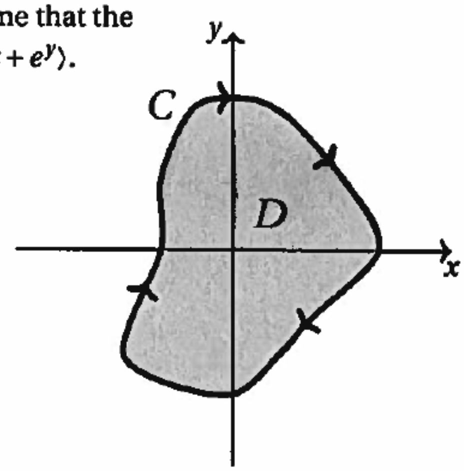
$$= \int_0^1\int_y^2 2y-1 \quad dx\, dy$$

$$= \int_0^1 \underbrace{(2y-1)(2-y)}_{-2y^2+5y-2}\, dy = -\frac{2}{3}y^3 + \frac{5}{2}y^2 - 2y \Big|_{y=0}^{1}$$

$$= -\frac{2}{3} + \frac{5}{2} - 2 = \frac{-4+15-12}{6} = -\frac{1}{6}$$

Question 2

Let C be the oriented curve shown at right, oriented **clockwise**. Assume that the region D bounded by C has area 6. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F} = \langle \sin x, 2x + e^y \rangle$.
(4 points)



By Green's theorem, since $C = -\partial D$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D (Q_x - P_y)\, dA$$

$$= -\iint_D 2\, dA$$

$$= -2 \iint_D dA$$

$$= -2\,(\text{Area}(D))$$

$$= -12$$

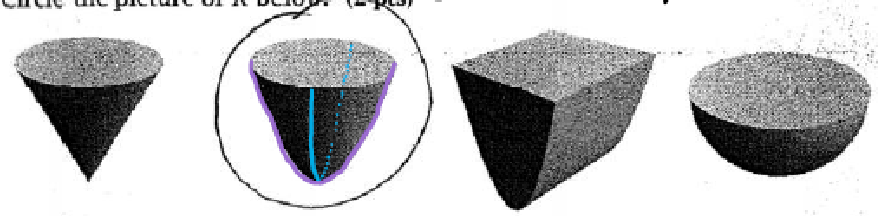
$\int_C \mathbf{F} \cdot d\mathbf{r} =$ $\boxed{-12}$

Question 3

Consider the region R in $\mathbb{R}^3$ above the surface $x^2 + y^2 - z = 4$ and below the xy-plane. Also consider the vector field $\mathbf{F} = (0, 0, z)$.

$$\left.\begin{matrix} x=0: & z = y^2 - 4 \\ y=0: & z = x^2-4 \end{matrix}\right\} \text{parabolas}$$

(a) Circle the picture of R below. (2 pts)



(b) Directly calculate the flux of **F** through the entire surface ∂R, with respect to the outward unit normals. **(10 pts)**

Now $\partial R = (T = \;)$ and $(S = \;)$. First, $\iint_T \vec{F}\cdot\vec{n}\,dA = \iint_T (0,0,0)\cdot(0,0,1)\,dA = \iint_T 0\,dA = 0$. Second, lets param. S by $\vec{r}(u,v) = (u, v, u^2+v^2-4)$ on $D = \{u^2+v^2 \le 4\}$. Then

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1).$$

As this points the wrong way we use $\vec{r}_v \times \vec{r}_u$ instead. Now Flux $= \iint_S (\vec{F}\cdot\vec{n})\,dA =$

$$\iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_v\times\vec{r}_u)\,du\,dv = \iint_D (0,0,u^2+v^2-4)\cdot(2u,2v,-1)\,du\,dv$$

$$= \iint_D 4-u^2-v^2\,du\,dv = \int_0^2\int_0^{2\pi}(4-r^2)r\,d\theta\,dr = 2\pi\int_0^2 4r - r^3\,dr$$

$$= 2\pi\left(2r^2 - \frac{r^4}{4}\right)\Big|_{r=0}^{r=2} = 2\pi(8-4) = \boxed{8\pi}$$

Hence $\iint_{\partial R}\vec{F}\cdot\vec{n}\,dA = \iint_T \vec{F}\cdot\vec{n}\,dA + \iint_S \vec{F}\cdot\vec{n}\,dA = 0 + 8\pi = \boxed{8\pi}$

(c) Use the Divergence Theorem and your answer in (b) to compute the volume of R. **(3 pts)**

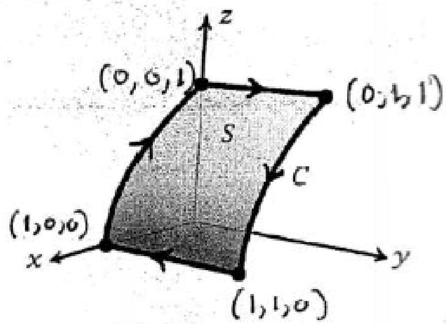
$$\iint_{\partial R}\vec{F}\cdot\vec{n}\,dA = \iiint_R \operatorname{div}\vec{F}\,dV = \iiint 1\,dV = \text{Volume}.$$

$$\operatorname{div}\vec{F} = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 1$$

So: $\boxed{\text{Vol} = 8\pi.}$

Question 4

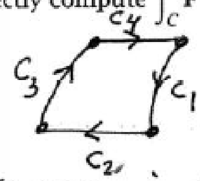
Let C be the curve shown at right, which is the boundary of the portion of the surface $x + z^2 = 1$ in the positive octant where additionally $y \le 1$.



(a) Label the four corners of C with their (x, y, z)-coordinates. **(1 pt)**

(b) For $\mathbf{F} = \langle 0, xyz, xyz \rangle$, directly compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(6 pts)**

Break C up into



Notice if <u>any</u> of x, y or z is 0, then $\vec{F} = \vec{0}$. Thus for any C_i except C_1, we have $\int_{C_i} \vec{F} \cdot d\vec{r} = \int_{C_i} \vec{0} \cdot d\vec{r} = 0$.

For C_1 we parameterize $-C_1$ via

$$\vec{r}(t) = (1 - t^2, 1, t) \quad \text{for } 0 \le t \le 1.$$

So
$$\int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{-C_1} \vec{F} \cdot d\vec{r} = -\int_0^1 (0, t - t^3, t - t^3) \cdot (-2t, 0, 1)\, dt$$
$$= \int_0^1 t^3 - t\, dt = \left. \frac{t^4}{4} - \frac{t^2}{2} \right|_{t=0}^{1} = -\frac{1}{4}$$

Hence $\int_C \vec{F} \cdot d\vec{r} = \sum \int_{C_i} \vec{F} \cdot d\vec{r} = 0 + 0 + 0 - \frac{1}{4} = \boxed{-\frac{1}{4}}$

(c) Compute $\operatorname{curl} \mathbf{F}$. **(2 pts)**

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xyz & xyz \end{vmatrix} = (xz - xy, -yz, yz)$$

(d) Use Stokes' Theorem to compute the flux of $\operatorname{curl} \mathbf{F}$ through the surface S where the normals point out from the origin. **(3 pts)**

$$\iint (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA = -\int_C \vec{F} \cdot d\vec{r} = \frac{1}{4}$$

since orient of C doesn't mesh with $\vec{n}$.

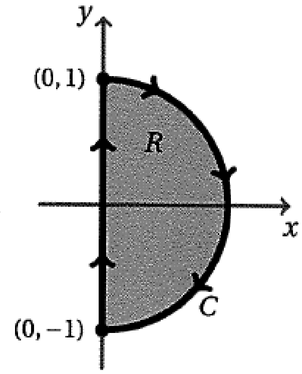
(e) Give two distinct reasons why the vector field $\mathbf{F}$ is *not* conservative. **(2 pts)**

$\operatorname{curl} \vec{F} \neq 0$ and C is a closed curve with $\int_C \vec{F} \cdot d\vec{r} \neq 0$.

Question 5

Consider the vector field $\mathbf{F}(x,y) = \langle ye^x,\ e^x + x\rangle$. Let R be the half disk below, and let C be the boundary of R, oriented as shown.

P Q



(a) Use Green's Theorem to compute $\int_C \mathbf{F}\cdot d\mathbf{r}$. **(4 points)**

$$\int_C \vec{F}\cdot dr = -\int_{-C} \vec{F}\cdot d\vec{r} = -\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$= -\iint_R (e^x+1)-(e^x)\, dA = -\iint_R 1\, dA$$

$$= -\text{Area}(A) = -\pi/2$$

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = -\pi/2}$$

(b) Let C_0 be the round part of C, that is, just the semicircle from $(1,0)$ to $(-1,0)$, not including the y-axis. Compute $\int_{C_0} \mathbf{F}\cdot d\mathbf{r}$. **(2 points)**

Let A be the vertical segment from $(0,-1)$ to $(0,1)$. Then $\int_C \vec{F}\cdot dr = \int_{C_0} \vec{F}\cdot dr + \int_A \vec{F}\cdot d\vec{r}$.

Now $\int_A \vec{F}\cdot dr = \int_{-1}^{1} \langle y, 1\rangle \cdot \langle 0, 1\rangle\, dt = \int_{-1}^{1} 1\, dt = 2$

where we have used the param. Thus:

$\vec{r}(t) = (0, t)$ for $-1 \le t \le 1$

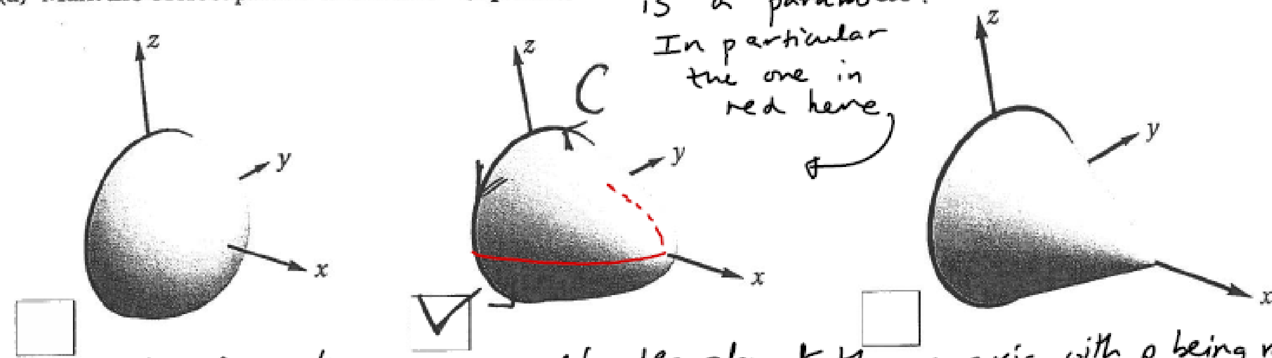
$$\boxed{\int_{C_0} \mathbf{F}\cdot d\mathbf{r} = -\pi/2 - 2}$$

Question 6

Let S be the surface in $\mathbb{R}^3$ parameterized by $\mathbf{r}(u, v) = \langle 2 - 2v^2,\ v\cos u,\ v \sin u\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$.

(a) Mark the correct picture of S below. **(2 points)**

Note that $r(0, v) = (2-2v^2, v, 0)$ is a parabola. In particular the one in red here.



also in cylindrical coordinates about the x-axis, with ρ being radius, satisfies the equation of the paraboloid $x = 2 - 2\rho^2$.

(b) For the vector field $\mathbf{F} = \langle 0, -z, y\rangle$, directly evaluate $\iint_S (\operatorname{curl} F) \cdot \mathbf{n}\, dA$ where $\mathbf{n}$ is unit normal vector field that points in the positive x-direction. **(5 points)**

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -z & y \end{vmatrix} = \langle 2, 0, 0\rangle$$

points in direction of $-\vec{n}$.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 0 & -v\sin u & v\cos u \\ -4v & \cos u & \sin u \end{vmatrix} = \langle -v, -4v^2 \cos u, -4v^2 \sin u\rangle$$

$$\iint_S (\operatorname{curl} \vec{F}) \vec{n}\, dA = \int_0^{2\pi}\int_0^1 \langle 2,0,0\rangle \bullet \langle +v, 4v^2\cos u, 4v^2 \sin u\rangle\, dv\, du$$

$$= \int_0^{2\pi}\int_0^1 2v\, dv\, du = \int_0^{2\pi} v^2\Big|_{v=0}^{1} du$$

$$= \int_0^{2\pi} 1\, du = 2\pi$$

$$\boxed{\iint_S (\operatorname{curl} F)\cdot \mathbf{n}\, dA = \quad 2\pi}$$

(c) Check your answer in (b) using Stokes' Theorem. **(3 points)**

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, -\sin t, \cos t\rangle \bullet \vec{r}\,'(t)\, dt$$

$\vec{r}(t) = (0, \cos t, \sin t)$ $\qquad = \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt$

$\vec{r}\,'(t) = (0, -\sin t, \cos t)$ $\qquad = \int_0^{2\pi} 1\, dt = \boxed{2\pi.}$ ✓