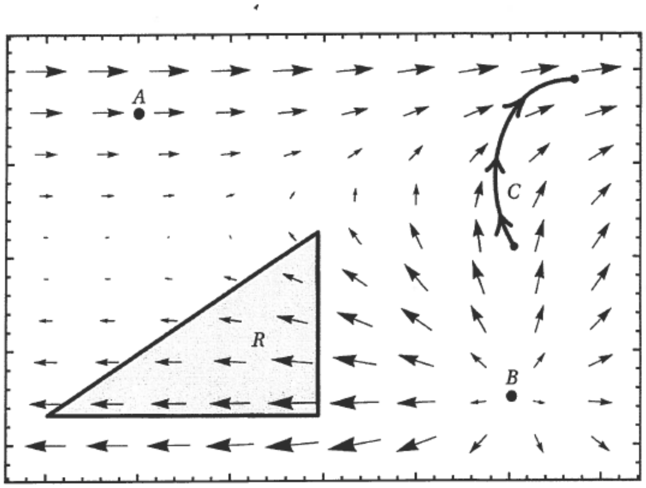
Math 241 Solutions Exam 4 Practice 2 **Fall 2024**

Question 1

Consider the vector field **F** on $\mathbb{R}^2$ shown below right. For each part, circle the best answer. **(1 point each)**



(a) The line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is:

negative zero [✓ positive]

(b) At A, the vector curl**F** is:

$\langle 1,0,0\rangle$ [✓ $\langle 0,0,-1\rangle$] $\langle 0,0,1\rangle$

(c) At B, the divergence div**F** is:

negative zero [✓ positive]

(d) If $\mathbf{F}=\langle P,Q\rangle$, then $\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,dA$ is:

[✓ negative] zero positive

(e) The vector field **F** is conservative: True [✓ False]

(a) The vectors in **F** point in the same general direction as the curve C (the angle between **F** and the direction of C is acute). Thus the dot product $\mathbf{F}\cdot\mathbf{dr}$ is positive, so the integral is positive.

(b) The curl of a vector field living in $\mathbb{R}^2$ (two-dimensions) is always in the z direction, so it cannot be $\langle 1,0,0\rangle$.

If $\mathbf{F}=\langle P,Q\rangle$, note that Q is always 0 in that part of the graph, so $\frac{\partial Q}{\partial x}=0$. Since P is increasing as y increases, $\frac{\partial P}{\partial y}$ is positive. Thus:

$$\begin{aligned} z\text{-coord of curl}\,(\mathbf{F}) &= \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y} \\ &= 0-(\text{positive}) \\ &= \text{negative} \end{aligned}$$

The only answer with a negative z-coordinate is $\langle 0,0,-1\rangle$.

Another way to answer this is by the right-hand rule. If you curl your fingers in the direction **F** is rotating, your thumb points in the direction of curl (**F**). **F** is turning clockwise, so your thumb will point into the page (which is in the negative z direction. (It may not look like it's rotating since the vectors all point horizontally, but consider this: let **F** be the flow of water, and drop a stick in the water at A. Then the stick will turn clockwise since the top is being pushed harder than the bottom.)

(c) P goes from negative to positive as you move in the x direction, so P is increasing. Thus $\frac{\partial P}{\partial x}$ is positive. Also, Q goes from negative to positive as you move in the y direction, so $\frac{\partial Q}{\partial y}$ is positive. Then $\text{div}\,(\mathbf{F})=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}$ is positive.

Intuitively, **F** flows away from B, so B is acting as a source, which means the divergence is positive.

(d) The field is turning clockwise in R (by right-hand rule; see explanation for part (b) to see why), so the curl is pointing in the negative z-direction. Since $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ is the z-coordinate of curl, $\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}$ is negative and so the integral is negative.

(e) Conservative vector fields always have zero curl, and **F** has non-zero curl. (For example, no matter what your answer for (b) is, it's not zero.)

Another reason is that since $\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,\mathrm{d}A$ is non-zero, and $\iint_R \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\,\mathrm{d}A = \oint_{\partial R}\mathbf{F}\cdot\mathbf{dr}$ by Green's Theorem, then **F** is not path-independent since $\oint_{\partial R}\mathbf{F}\cdot\mathbf{dr}$ is non-zero.

Question 2

Consider the surfaces S and H show below right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. For each part, circle the best answer.

a) For $\mathbf{F} = \langle yz,\ xz + x,\ z\rangle$, the integral $\iint_H \mathbf{F}\cdot\mathbf{n}\ dA$ is:

negative zero [✓ positive] **(1 point)**

b) The flux of $\operatorname{curl}\mathbf{F} = \langle -x, y, 1\rangle$ through H is:

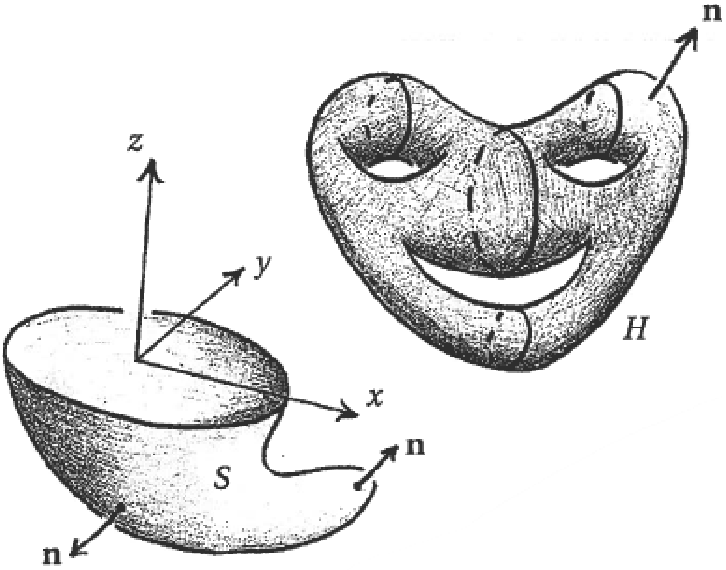
negative [✓ zero] positive **(1 point)**

c) The integral $\iint_S (\operatorname{curl}\mathbf{F})\cdot d\mathbf{S}$ is:

-2π [✓ $-\pi$] 0 π 2π **(2 points)**

(d) For $\mathbf{G} = \langle y,\ z,\ 2\rangle$, the integral $\iint_S \mathbf{G}\cdot\mathbf{n}\ dA$ is:

[✓ -2π] $-\pi$ 0 π 2π **(2 points)**



Explanations on next page.

Q8 Continued.

(a) We have $\operatorname{div}(\mathbf{F}) = 0+0+1 = 1$. Let R be the three-dimensional inside of the surface H. By the Divergence Theorem, $\iint_H \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}A = \iiint_R \operatorname{div}(\mathbf{F}) \, \mathrm{d}V = \iiint_R 1 \, \mathrm{d}V = \text{Volume}(R) > 0$.

(b) By Stokes' Theorem, $\iint_H \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial H} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. But H has no boundary, so the integral of anything over the boundary of H is zero.

(c) By Stokes' Theorem, $\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. The boundary of S is the unit circle in the xy-plane; the normal vector of S is downward, so the circle is oriented clockwise (as viewed from above), by the right-hand rule.

There are two options to go from here. The first: by Stokes' Theorem, any other surface with the same boundary and orientation has the same flux integral of $\operatorname{curl}(\mathbf{F})$. The unit disk in the xy-plane, pointing downwards, is one such surface. Then:

$$\begin{aligned}
\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \iint_{\text{disk}} \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} \\
&= \iint_{\text{disk}} \langle -x, y, 1 \rangle \cdot \langle 0, 0, -1 \rangle \, \mathrm{d}A \\
&= \iint_{\text{disk}} -1 \, \mathrm{d}A \\
&= -\text{Area}(\text{disk}) \\
&= -\pi
\end{aligned}$$

The other option is to calculate the line integral over the boundary directly. $\mathbf{r}(t) = \langle \sin t, \cos t, 0 \rangle$, $0 \le t \le 2\pi$ is a parametrization of the circle, going clockwise. Also, $\mathbf{r}'(t) = \langle \cos t, -\sin t, 0 \rangle$. Then:

$$\begin{aligned}
\iint_S \operatorname{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} &= \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, \mathrm{d}t \\
&= \int_0^{2\pi} \langle (\cos t)(0), (\sin t)(0) + \sin t, 0 \rangle \cdot \langle \cos t, -\sin t, 0 \rangle \, \mathrm{d}t \\
&= \int_0^{2\pi} -\sin^2(t) \, \mathrm{d}t \\
&= -\pi
\end{aligned}$$

(d) Make S into a solid three-dimensional shape R by adding the unit disk, oriented upwards, so that the orientation of the boundary ∂R is outwards. Then $\iint_{\partial R} =$

Question 3

Let $\mathbf{F}(x, y, z) = \left\langle \frac{x^3}{3},\ x^2\cos(z) + \frac{y^3}{3},\ \frac{z^3}{3} \right\rangle$ and let S denote the surface defined by $x^2 + y^2 + z^2 = 1$, equipped with the inward-pointing unit normal vector field $\mathbf{n}$. Compute $\iint_S \mathbf{F}\cdot\mathbf{n}\,\underset{dA}{dS}$ by any valid method. **(6 points)**

$$\iint_S \vec{F}\cdot\vec{n}\, dA \underset{\text{divergence Thm}}{=} \overset{\text{inward pointing}}{-}\iiint_{Ball} \operatorname{div}\vec{F}\, dV = -\iiint_{Ball} \underbrace{x^2+y^2+z^2}_{\rho^2}\, dV$$

$$\underset{\text{change to Spherical}}{=} -\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= -\int_0^{2\pi}\int_0^{\pi} \sin\phi\, \left.\frac{\rho^5}{5}\right|_{\rho=0}^{1} d\phi\, d\theta$$

$$= -\frac{1}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta = +\frac{1}{5}\int_0^{2\pi} \cos\phi\Big|_{\phi=0}^{\phi=\pi} d\theta$$

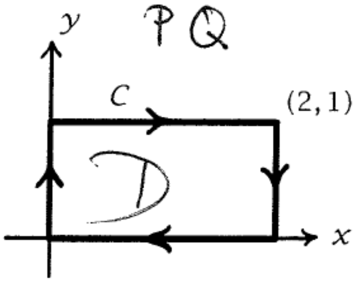
$$= -\frac{2}{5}\int_0^{2\pi} 1\, d\theta = -\frac{4\pi}{5}$$

$$\boxed{\iint_S \mathbf{F}\cdot\mathbf{n}\,\underset{dA}{\cancel{dS}} = -\frac{4\pi}{5}}$$

Question 4

Let C be the oriented curve in $\mathbb{R}^2$ shown at right. For the vector field $F(x, y) = (x^3, x^2)$, use Green's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(6 points)**

P Q



$$\int_C F\cdot d\vec{r} = -\int_{\underbrace{-C}_{\text{C reversed so it's anti-clockwise}}} \vec{F}\cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

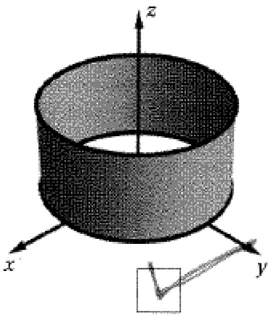
$$= -\int_0^1\int_0^2 2x \quad dx\, dy = -\int_0^1 x^2\Big|_{x=0}^{2} dy$$

$$= -\int_0^1 4\, dy = \boxed{-4}$$

Question 5

Consider the surface S parameterized by $\mathbf{r}(u,v) = \langle \cos u, \sin u, v\rangle$ for $0 \le u \le 2\pi$ and $0 \le v \le 1$.

(a) Mark the picture of S below. **(2 points)**



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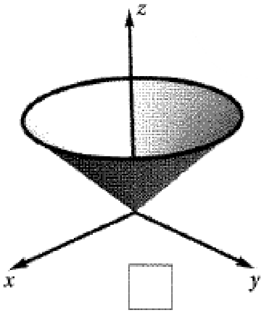

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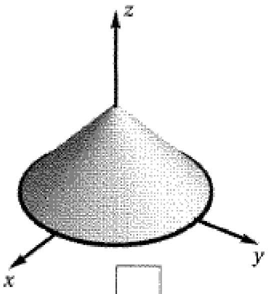

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(b) Consider the vector field $\mathbf{F} = \langle yz, -xz, 1\rangle$ which has $\operatorname{curl}\mathbf{F} = \langle x, y, -2z\rangle$. Directly evaluate $\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\, dS$ via the given parameterization, where $\mathbf{n}$ is the outward normal vector field. **(4 points)**

$$= \int_0^1\int_0^{2\pi} \langle \cos u, \sin u, -2v\rangle \bullet \overbrace{\langle \cos u, \sin u, 0\rangle}^{r_u \times r_v}\, dv\,du$$

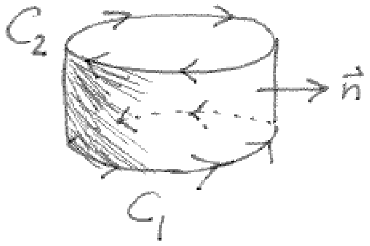
$$= \int_0^1\int_0^{2\pi} \overbrace{\cos^2 u + \sin^2 u + 0}^{1}\, dv\,du$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad = 2\pi$$

$$= \langle \cos u, \sin u, 0\rangle$$

$\iint_S (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}\, dS =$ 2π

(c) Check your answer in (b) using Stokes' Theorem. **(4 points)** Param for C_1: $\vec{r}(t) = (\cos t, \sin t, 0)$



$$\int_{C_1} \vec{F}\bullet d\vec{r} = \int_0^{2\pi} \langle 0,0,1\rangle \bullet \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$

$$\int_{C_2} \vec{F}\bullet d\vec{r} = \int_0^{2\pi} \overbrace{\langle \cos t, -\sin t, 1\rangle \bullet \langle \cos t, -\sin t, 0\rangle}^{\cos^2 t + \sin^2 t + 0 = 1}\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi.$$

Matches!

$\vec{r}_2(t) = (\sin t, \cos t, 1)$

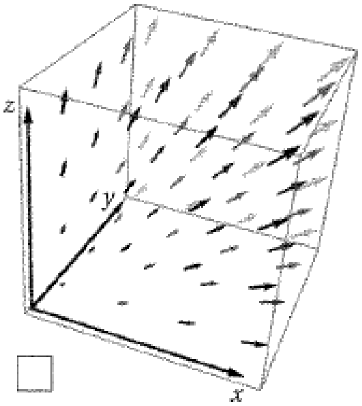
So flux $= \int_{\partial S} \vec{F}\bullet d\vec{r} = 0 + 2\pi = \boxed{2\pi}$

Question 6

For each problem, circle the best answer. **(1 point each)**

(a) Consider the vector field $\mathbf{F} = \langle 1,\ x,\ -z\rangle$. The vector field $\mathbf{F}$ is: conservative [✓ not conservative]

(b) Mark the plot of $\mathbf{F}$ on the region where each of x, y, z is in $[0, 1]$:



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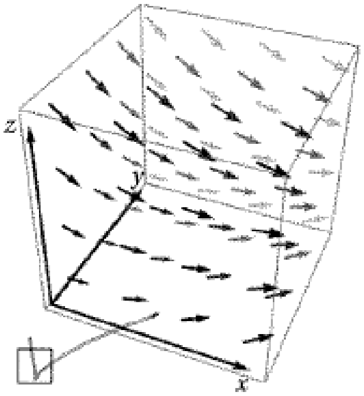

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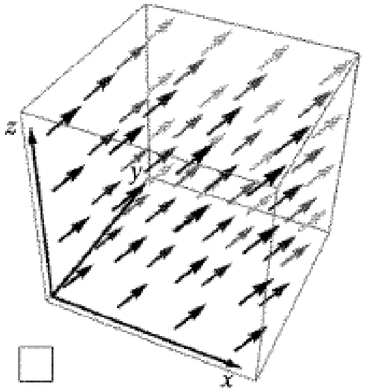

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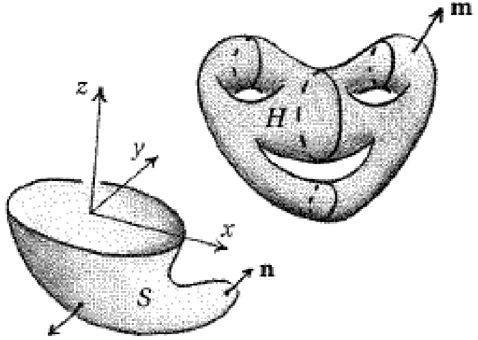
(c) For the leftmost vector field in part (b) is the divergence: negative zero [✓ positive]

Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. Let $\mathbf{G} = \langle x,\ y,\ z\rangle$.



(d) The flux $\iint_H \mathbf{G}\cdot\mathbf{m}\,dS$ is:

negative zero [✓ positive]

(e) The flux $\iint_S \mathbf{G}\cdot\mathbf{n}\,dS$ is: negative zero [✓ positive]

(f) The flux $\iint_S (\operatorname{curl}\mathbf{G})\cdot\mathbf{n}\,dS$ is: negative [✓ zero] positive

Explanations on next page.

Q12 Continued.

Answer:

(a) If the vector field $\mathbf{F}$ were conservative, we would have $\operatorname{curl}\mathbf{F} = 0$. Since $\operatorname{curl}\mathbf{F} \neq 0$, we have that $\mathbf{F}$ is not conservative.

(b) The option on the right is a constant vector field, which is inconsistent with $\mathbf{F}$. The option on the left has a changing x-component, which is again inconsistent with $\mathbf{F}$.

(c) If we imagine this vector field as describing the flow of a fluid, we see that it is flowing out of the origin, which means the divergence is positive.

(d) By Divergence Theorem, this integral calculates $3V$, where V is the volume of the interior of H.

(e) Let D be the unit disk in the xy-plane, and let E be the region bounded by D and S. Divergence Theorem tells us

$$\iint_S \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}S = 3\iiint_E \mathrm{d}V - \iint_D \mathbf{G}\cdot\mathbf{n}\,\mathrm{d}S.$$

Now the last integral in this equation is zero since, on D, the normal vector is perpendicular to $\mathbf{G}$.

(f) Since $\operatorname{curl}\mathbf{G} = 0$, this integral is zero.